Topology Preliminary Exam

January 18, 1994

DO AS MANY PROBLEMS AS YOU CAN.

1. Let X be a complete metric space.

   (a) Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ be a sequence of non-empty, closed, bounded subsets of X, whose diameters $\to 0$. Prove that $\bigcap_{m=1}^{\infty} F_m$ is a single point.

   (b) Prove that X is a Baire space: The intersection of a countable family of open dense subsets of X is dense in X.

2. (a) Carefully state Urysohn's Lemma.

   (b) Show that a subspace of a normal space is completely regular. (X is completely regular if one-point sets are closed and for every $a \in X$, every closed $B \subset X$, $a \notin B$, there exists a continuous function $f : X \to [0,1]$ such that $f(a) = 0$, $f(B) = 1$.)

3. Let X be connected, $f, g : X \to [0,1]$ continuous, f surjective. Prove that there exists $x \in X$ with $f(x) = g(x)$.

4. Prove that $X \times Y$ is compact $\iff$ X, Y are compact.

5. Show that $\mathbf{R}^2$ minus a countable set is path connected.

6. Let the wedge $S^1 \vee S^1$ be embedded in $S^1 \times S^1$ via $S^1 \times \{(1,0)\} \cup \{(0,1)\} \times S^1 \subset S^1 \times S^1$. The quotient space $S^1 \times S^1 / S^1 \vee S^1$ is homeomorphic to a familiar space. Find this space and prove they are homeomorphic.

7. (a) Prove that there does not exist a retraction $f : D^2 \to S^1$.

   (b) Show that every continuous $g : D^2 \to D^2$ has a fixed point.

8. (a) Compute the fundamental group of $X = \mathbf{R}P^2 \vee S^1$, the one-point union of a projective plane and a circle.

   (b) X has precisely 3 distinct (connected) 2-fold covering spaces. Find them. Also describe them algebraically, i.e. in terms of surjections $\pi_1\ X \to \mathbf{Z}_2$.