

Do as many problems as you can.

1. Let X be a topological space and let A be a dense subset of X . Let Y be a Hausdorff space, and let $g, h: X \rightarrow Y$ be continuous functions which agree on A . Prove that $g = h$.
2. Let $f: X \rightarrow Y$ be a continuous function, and let $G \subset X \times Y$ be its graph, that is, the subset $G = \{(x, y) \mid y = f(x)\}$. If Y is Hausdorff, prove that G is closed in $X \times Y$.
3. Prove the following partial converse to Question 3: If Y is compact and $f: X \rightarrow Y$ is a map whose graph is closed in $X \times Y$, then f is continuous.
4. Let A be a connected subset of a topological space X . Show that the closure of A is connected.
5. State and prove the Urysohn metrization theorem.
6. Recall that the *lexicographic order topology* on the unit square $I^2 = [0, 1] \times [0, 1]$ is generated by the subbase consisting of the subsets

$$S_{a,b,c,d} = \{a\} \times (b, 1] \cup (a, c) \times [0, 1] \cup \{c\} \times [0, d)$$

for all quadruples of numbers a, b, c, d with $0 \leq a < c \leq 1$, $0 < b < 1$, $0 < d < 1$, and the subsets

$$T_{k,l,m} = \{k\} \times (l, m)$$

for all triples k, l, m with $0 \leq k \leq 1$, $0 \leq l < m \leq 1$. We denote the unit square with this topology by I_ℓ^2 .

(a) Show that I_ℓ^2 is compact.

(b) Show that I_ℓ^2 is not second countable.

(Note: all compact metric spaces are second countable, so this shows I_ℓ^2 is not metrizable.)

7. Suppose that X is path-connected and let $h: X \rightarrow Y$ be a continuous map with $h(x_0) = y_0$ and $h(x_1) = y_1$. Recall that h induces a group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$, which will be denoted by $(h_{x_0})_*$, and also a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(Y, y_1)$, which will be denoted by $(h_{x_1})_*$. Prove that there are isomorphisms $\phi: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ and $\psi: \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_1)$ so that

$$\psi \circ (h_{x_0})_* = (h_{x_1})_* \circ \phi.$$

8. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$, the wedge of two projective planes.
 - (a) Find $\pi_1(X)$.
 - (b) Describe the universal cover of X and the action of $\pi_1(X)$ by the covering transformations.
 - (c) Does $\pi_1(X)$ have elements of infinite order? Explain.