PRELIMINARY EXAM

TOPOLOGY

AUGUST 2007

1) Suppose $p : E \to B$ is a covering map and $f, g : [0,1] \to B$ are continuous maps such that $pf = pg$ and $f(0) = g(0)$. Prove that $f(t) = g(t)$ for all t in $[0,1]$.

2) Let X be a 2^{nd} countable space.

   a) Prove that at most countably many points of X can be isolated points.

   b) Prove that every open cover of X has a countable subcover.

3) a) Let $r : X \to A$ be a retraction and let $i : A \to X$ be the inclusion map and let r^* and i^* be the induced homomorphisms of fundamental groups with some common base point. Assuming functorial properties of induced maps, what can you say about r^* and i^*?

Now using part a as needed, tell whther each of the following is true or false and justify your answer. You can also assume the fundamental groups of specific spaces are known.

   b) If A is a retract of X and A is simply connected, so is X.

   c) If A is a retract of X and X is simply connected, then so is A.

   d) There is no retract of the projective plane P^2 onto a subspace which is homeomorphic to the circle S^1.

   e) There is a retraction of the closed unit disk in R^2 onto its boundary.

4) Let I_P^ω and I_B^ω denote the countable product of unit intervals with the product and box topologies, respectively.

   a) Determine with proof whether the identity map from I_P^ω to I_B^ω is continuous or not.

   b) Same question for the identity map from I_B^ω to I_P^ω.

   c) Let A denote the set of points which are zero in all but finitely many coordinates. Determine, with proof, the closure of A in each topology.

5) Recall that a surjective map $f : X \to Y$ is a quotient map provided that a set U is open in Y if and only if $f^{-1}(U)$ is open in X.

   a) Suppose $p : X \to Y$ and $q : Y \to X$ are continuous maps and $p \circ q =$ the identity map of Y. Prove that p is a quotient map.

   b) Let $f : R \times R \to R$ be projection onto the first coordinate and let p be the restriction of f to the space X consisting of all points (x, y) with $x \geq 0$ or $y = 0$ (or both). Prove that p is a quotient map that is neither an open map nor a closed map.

6) Let $X = P^2 \vee S^1$ denote the one point union of the projective plane and a circle, and let Y denote the subset of R^3 that is the union of the sets A, B, C where

$$A = \{(x, y, z) | x^2 + y^2 \leq 1 \text{ and } z = 0 \text{ or } 1\}$$
$$B = \{(-\frac{1}{2}, 0, z) | 0 \leq z \leq 1\}$$
$$C = \{(\frac{1}{2}, 0, z) | 0 \leq z \leq 1\}$$

   a) Find the fundamental group of X and justify your answer.

   b) Find the fundamental group of Y and justify your answer.

   c) Pick one of these two spaces and describe its universal covering space.

7) Suppose X and Y are arbitrary spaces with Y compact. Let x_0 be a point of X and let U be an open set in the product space $X \times Y$ that contains $x_0 \times Y$. Prove that there exists an neighborhood V of x_0 such that $V \times Y$ is contained in U.