Topology Preliminary Examination **January, 2005**

Do as many problems as you can.

1. Let X be a compact space, and suppose there is a finite family of continuous functions $f_i \colon X \to \mathbb{R}$, $i = 1, \dots, n$, with the following separation property: Given $x \neq y$ in X there is an i such that $f_i(x) \neq f_i(y)$. Prove that X is homeomorphic to a subset of $\mathbb{R}^n$.

2. Let $f \colon X \to Y$ be a continuous function, and let $G \subset X \times Y$ be its graph, that is, the subset $G = \{(x, y) \mid y = f(x)\}$. If Y is Hausdorff, prove that G is closed in $X \times Y$.

3. (a) State Urysohn's lemma and Tietze's extension theorem.

   (b) Let X be a metric space and let $\{x_n\}$, $n \geq 1$, be an infinite sequence in X such that $d(x_i, x_j) \geq 1$ for $i \neq j$. Prove that there is a continuous map $f \colon X \to \mathbb{R}$ such that $f(x_n) = n^2$ for all n.

4. Let G be the wedge of infinitely many unit intervals indexed by positive integers. That is, the set G is the union of $[0, 1]_i$, $i = 1, 2, 3, \dots$, with all 0-endpoints identified. Recall that the *weak topology* on G is the smallest collection of subsets such that the intersection with each closed edge is open within that edge. Prove that G with the weak topology is not metrizable.

5. The *connected components* of a topological space X are the equivalence classes for the following relation: $x \sim y$ if and only if there is a connected subset of X containing both x and y.

   (a) Prove this relation $\sim$ is indeed an equivalence relation.

   (b) Prove that every connected component is connected.

6. Let α and β be paths from x_0 to x_1 in a space X. If $\pi_1(X, x_0) = 0$, prove that $\alpha \simeq_p \beta$.

7. Let $p \colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Suppose the following condition holds: For every nontrivial element $[\alpha] \in \pi_1(X, x_0)$ and every representative loop α, the lift $\tilde{\alpha}$ starting at $\tilde{x}_0$ is not a loop. Prove that $\pi_1(\widetilde{X}, \tilde{x}_0) = 0$.

8. (a) Compute the fundamental group of $S^1 \vee S^2 \vee S^3$, the one-point union of the circle, the two-dimensional sphere, and the three-dimensional sphere.

   (b) Find the universal cover of $S^1 \vee S^2 \vee S^3$.