Topology Preliminary Examination **August 29, 2003**

1. Prove that $f: X \to Y$ is continuous if, and only if, for all $A \subset Y$,

$$\overline{f^{-1}(A)} \subset f^{-1}(\overline{A}).$$

2. Show that $\mathbb{R}^J$ is not metrizable if J is uncountable.

3. Find a subset of $\mathbb{R}^2$ that is path connected but not locally path connected at any of its points. (Hint: First find a set that is path connected but locally connected at only one of its points.) Prove your assertion.

4. Show that a locally compact Hausdorff space is regular.

5. Let C_n be the subspace of $\mathbb{R}^2$ defined by $C_n = \{(x, y) | (x - \frac{1}{n})^2 + y^2 = (\frac{1}{n})^2\}$. Let Y be the subspace

$$Y = \bigcup_{n \in \mathbb{Z}_+} C_n$$

of $\mathbb{R}^2$, and let X be the subspace $C_1 \times \mathbb{Z}_+$ of $\mathbb{R}^2 \times \mathbb{R}$. Define $g: X \to Y$ by $g(((x, y), n)) = (\frac{x}{n}, \frac{y}{n})$. Show that g is continuous and surjective, but g is not a quotient map.

6. State and prove the Urysohn Metrization Theorem.

7. Let $p: (E, e) \to (B, b)$ be a covering space of path connected spaces.

   a) Show that $p_*: \pi_1(E, e) \to \pi_1(B, b)$ is a monomorphism.

   b) Let $B = T^2$, a torus. Find 4 distinct covering spaces of T^2 corresponding to different subgroups of index 6 in $\pi_1(T^2)$.

8. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$, the wedge of two projective planes.

   a) Find $\pi_1(X)$.

   b) Describe the universal cover of X, and the covering transformations.

   c) Does $\pi_1(X)$ have elements of infinite order? Explain.