**Topology Preliminary Exam** **June 3, 2003**

Do as many problems as you can.

1. Let $X = \mathbb{R}^2/\{y\text{-}axis\}$, the plane with the y-axis collapsed to a point, with the quotient topology.

   (a) Is $X$ Hausdorff? Support your answer.

   (b) Is $X$ locally compact? Support your answer.

2. Prove that a compact Hausdorff space is normal.

3. Let $X$ be a compact metric space with metric $d$ and the property that for all $t < 1$, there are pairs of points $x_t$, $y_t$ so that $d(x_t, y_t) = t$. Prove there are points $x$ and $y$ so that $d(x, y) = 1$.

4. Let $\sim$ be the equivalence relation on the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

   given by

$$(x, y, z) \sim (-x, -y, z).$$

   (In other words, each point is equivalent to the opposite point on the same circle of constant latitude.) Prove that $S^2/\sim$ with the quotient topology is homeomorphic to $S^2$.

5. (a) Prove that a compact metric space is second countable.

   (b) Let $G$ be a graph (that is, one-dimensional simplicial complex) having a vertex which is an endpoint of infinitely many edges. Recall that the *weak topology* on $G$ is the smallest collection of subsets such that the intersection with each open edge is open within that edge. Show that $G$ with the weak topology is not metrizable.

6. (a) Give an example of a space which is connected but is not path-connected. You need to explicitly describe the space and prove it is an example.

   (b) Prove that a CW complex is connected if and only if it is path-connected.

7. Let $p\colon \tilde{X} \to X$ be a covering map with $\tilde{X}$ and $X$ path-connected. Show that the number of points in $p^{-1}(x)$ is independent of $x$ in $X$ and equals the index of the subgroup $p_*\pi_1(\tilde{X})$ in $\pi_1(X)$. This number is called the number of *sheets* of the covering.

8. (a) Compute the fundamental group of $\mathbb{R}P^2 \vee S^1$, the one-point union of the projective plane and a circle.

   (b) Find all 2-sheeted and 3-sheeted coverings of $\mathbb{R}P^2 \vee S^1$.

   (c) Find the universal cover of $\mathbb{R}P^2 \vee S^1$.