The University at Albany

Department of Mathematics and Statistics

Ph.D. Program

Preliminary Examination in Real Analysis

Wednesday, June 3, 1998

1. State the following theorems:

   A. Lebesgue dominated convergence theorem.

   B. Fatou's lemma.

   C. Fubini's theorem.

   D. Egoroff's theorem.

   E. Radon-Nikodym theorem.

2. State and prove from first principles the Lebesgue monotone convergence theorem (in particular do not assume Fatou's lemma.)

3. Define what it means for a function, f, to be a Lebesgue measurable function. Prove that the sum of two Lebesgue measurable functions is a Lebesgue measurable function.

4. Prove that a real valued continuous function whose domain is a compact subset of the real numbers is uniformly continuous.

5. Let $f_n(x)$ be Lebesgue integrable functions on $\mathbf{R}$ such that $\displaystyle\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |f_n(x)|dx < \infty$. Prove that $f_n(x) \to 0$ almost everywhere as $n \to \infty$.

6. Let $\{a_{n,k}\} \subset \mathbf{R}$ with $|a_{n,k}| \leq 1$ for $n, k = 1, 2, 3, \ldots$. Suppose that for each n, $\lim_{k\to\infty} a_{n,k} = 0$. Let $p > 1$. Show that

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{a_{n,k}}{n^p} = 0 \ .$$

7. Let $f_n(x)$ be Lebesgue integrable functions on $[0,1]$ such that for each $n = 1, 2, 3, \ldots$ $\|f_n\|_1 = 1$. Suppose that the Lebesgue measure of the support of f_n tends to zero as n tends to infinity. Let $p > 1$: Show that $\|f_n\|_p$ tends to infinity as n tends to infinity.

8. A. State the Vitali Covering Theorem.

   B. Let $f(x)$ be a monotone increasing continuous function on $[0,1]$. Show that the set of x where $f'(x) = \infty$ is a set of Lebesgue measure zero. (Hint: Use part A.)

9. Let f and g be non-negative Lebesgue integrable functions on R. Prove that

$$\int_{-\infty}^{\infty} f(x)\ g(x)\ dx = \int_0^{\infty} \varphi(y)\ dy\ ,$$

where $\varphi(y) = \displaystyle\int_{A_y} g(x)\ dx$ and

$$A_y = \{x : f(x) > y\}\ .$$