Real Analysis Prelim. (August, 2007)

PART I: Do all of the following problems.

1. Show that the set of irrational numbers in the interval [0, 1] is uncountable. (Do not use the fact that [0, 1] itself is uncountable.!!)

2. State the following theorems:

a, Fatou's Lemma;
b, Egorov's Theorem;
c, Minkowski's Inequality;
d, Radon-Nikodym Theorem.

3. Prove 2(a), **and** show by example that the equality may not hold.

PART II: Do at least four of the following problems.

4. Show that $L^1([0,\ 1])$ with the metric defined by

$$\|f-g\|_1 = \int_{[0,\ 1]} |f-g|dm,$$

where m is the Lebesgue measure, is a complete metric space.

5. Prove 2(b).

6. Show that every Riemann integrable function $f:\ [a,\ b] \longrightarrow \mathbf{R}$ is Lebesgue integrable.

7. Assume $f:\ [a,\ b] \longrightarrow \mathbf{R}$ is absolutely continuous. Show that $f(E)$ has measure 0 for every $E \subset [a,\ b]$ of measure 0.

8. Let $\{f_n\}$ be a sequence in $L^2([0,\ 1])$ that converges to f in norm (i.e. $\lim_{n\to\infty} \|f_n - f\|_2 = 0$). Show that f_n converges in measure to f.

9. Let $0 \leq \lambda < 1$. Construct a closed subset E of $[0,\ 1]$ such that E has Lebesgue measure λ, and that the complement of E is dense in $[0,\ 1]$.

10. Let $f \in L^1(\mathbf{R})$, and for every $\lambda \geq 0$ let $E_\lambda = \{x \in \mathbf{R}:\ |f(x)| \leq \lambda\}$. Show that

$$\int_{\mathbf{R}} |f(x)|dx = \int_0^\infty m(E_\lambda)d\lambda,$$

where m is the Lebesgue measure on $\mathbf{R}$.