The University at Albany
Department of Mathematics and Statistics
Ph. D. Program
Preliminary Examination in Real Analysis
Thursday, August 26, 2004

Part I

Problem 1. State the following

a. Egorov's Theorem.

b. Radon Nikodym Theorem. Include the definition of absolute continuity for measures.

c. Fatou's Lemma.

d. Lebesgue Dominated Convergence Theorem.

e. Fubini's Theorem.

Problem 2.

a. Give the definition of Riemann Integral.

b. Give the definition of Lebesgue integral.

c. Give an example of a function in $[0, \infty)$ which is (improperly) Riemann integrable but not Lebesgue integrable.

PART II

Problem 3. Let f be integrable with respect to Lebesgue measure λ on $\mathbb{R}$. Let μ be the Borel measure on $\mathbb{R}$ defined by

$$\mu(I) = \int_I f(x)d\lambda(x) \qquad \text{for every open interval } I \text{ on} \mathbb{R}.$$

a. Prove $\mu(E) = \int_E f(x)d\lambda(x)$ for every Borel set $E \in \mathbb{R}$.

b. Show that $\mu << \lambda$.

c. Compute $d\mu/d\lambda$.

Problem 4. Let λ be the Lebesgue measure on $[0, 1]$. Recall that

$$\|f\|_\infty = inf\{M : \lambda(\{x : |f(x)| > M\}) = 0\}.$$

Prove that if $\|f\|_\infty < \infty$, then

a. $f \in L^p$ for all $p > 0$.

b.
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Problem 5. Show that if $f \in L^p$, then

$$\int |f(x)|^p d\lambda(x) = \int_0^\infty pt^{p-1}\lambda(\{x : |f(x)| > t\})\, dt.$$

Hint: one way to prove this is to write the right hand side as a double intergral.

Problem 6. $0 \le x, y \le 1$. Assume that for each such x, $f(x, y)$ is integrable as a function of y, and $\frac{\partial f}{\partial x}(x, y)$ is bounded. Show

a. $\frac{\partial f}{\partial x}(x, y)$ is measurable,

b. $\frac{\partial}{\partial x} \int_0^1 f(x, y)dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)dy$.

Problem 7. Let f be a function of bounded variation on $[0, 1]$. Show that

$$\frac{f(1/n) + f(2/n) + \cdots + f(n/n)}{n} \mapsto \int_0^1 f(x)dx.$$