The University at Albany

Department of Mathematics and Statistics

Qualifying Examination in Real Analysis for the Master's and Ph.D. Programs

Friday, June 6, 2003

PART I

Do problems 1 and 2.

1. State the following theorems:

   A. Lebesgue dominated convergence theorem.

   B. Fatou's lemma.

   C. Egoroff's theorem.

   D. Radon-Nikodym theorem.

   E. Hölder's inequality.

2. Assuming only the properties of Lebesgue measure:

   A. Give a definition of the Lebesgue integral.

   B. Use your definition to prove the Lebesgue monotone convergence theorem.

PART II

Do 6 of the following 8 problems.

3. A. Define what it means for a function whose domain is $[0, 1]$ to be measurable.

   B. Prove that if $f(x)$ is measurable so is $(f(x)^2)$.

4. Prove that the sum
$$\sum_{n=0}^{\infty} \int_0^{3/4} (1 - x^{1/3})^n \; x^2 \; dx$$
   converges to a finite limit and find its value.

5. Construct a non-measurable subset of $[0, 1]$. Justify in detail your construction.

6. A. Define Absolute Continuity on $[0, 1]$.

   B. Prove that an absolutely continuous function is continuous.

   C. Prove an an absolutely continuous function is of bounded variation.

   D. Why are absolutely continuous functions important in the Lebesgue theory?

7. Let $\{a_{n,k}\} \subset R$ with $|a_{n,k}| \leq 1$ for $n, k = 1, 2, 3, \ldots$. Assume that for each n, $\lim_{k\to\infty} a_{n,k} = 0$. Let $p > 1$. Show that $\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{n,k}/n^p = 0$.

8. A. State the Vitali covering theorem.

   B. Prove that if $f(x)$ is a monotone increasing continuous function for $x \in [0, 1]$ that the set of points where $f'(x) = \infty$ has measure 0. (Hint: use part A.)

9. Prove that if f and f_n are measurable functions on $[0, 1]$ and $f_n(x)$ converges to $f(x)$ almost everywhere then $f(x)$ converges in measure to $f(x)$.

10. Prove that if f and g are positive continuous functions on $(-\infty, \infty)$ which are periodic of period 1 that

$$\lim_{n\to\infty} \int_0^1 f(x)\ g(nx)dx = \int_0^1 f(x)dx \int_0^1 g(x)dx\ .$$