RANK ONE TRANSFORMATIONS WITH SINGULAR SPECTRAL TYPE

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Abstract

We show that a certain class of measures arising from generalized Riesz products are singular. In particular, cutting and stacking (i.e. rank one) transformations whose cuts do not grow too rapidly, have singular maximal spectral type. The precise condition is $\sum_{n=1}^{\infty}(1/w_n^2) = \infty$, where the w_n is the number of cuts at stage n.

* Supported by a Canada NSERC grant
** Supported by an NSF grant

§1. Introduction.

Let T be a rank one transformation on an interval. Such a transformation may be obtained inductively by the cutting and stacking method. For a detailed exposition of such constructions see Friedman [6]. This class of transformations has proven to be a rich source of examples in ergodic theory for transformations exhibiting different kinds of properties.

Rank one transformations are defined inductively on towers. A tower H is a collection of disjoint intervals of the same length, $\{I_i\}_{i=1}^{h}$, where T is defined on all but the last interval by just mapping linearly the interval I_i onto the next I_{i+1}, and h is called the height of H.

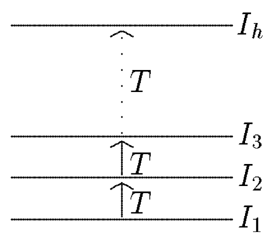

Figure 1: Rank one construction.

Start with the initial tower $H_0 = [0, 1]$ of height 1. Let H_n denote the n^{th} tower and h_n its height. Suppose H_n has already been defined. H_n consists of intervals of the same length which form a partition of $[0, r_n]$, for some $r_n > 1$, stacked one on top of the other in some order. To construct H_{n+1}, divide the tower H_n into w_n subcolumns of equal width. Then, on top of the k^{th} subcolumn, add a number $a_n(k)$ of consecutive disjoint intervals. The added intervals have the same width as the subcolumns of H_n and are taken to the right of r_n. That is, they form a partition of $[r_n, r_{n+1}]$, where $r_{n+1} = r_n + \sum_{k=1}^{w_n} a_n(k)\, l$, l =length of subcolumns of H_n. Then H_{n+1} is the column obtained by stacking these subcolumns one on top of the previous one, starting from the left.

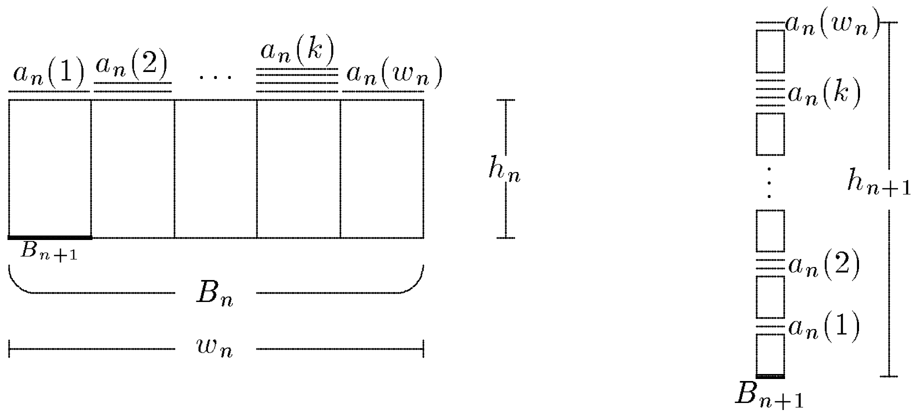

Figure 2: $(n+1)^{\text{th}}$–tower.

By construction, H_{n+1} has height

$$h_{n+1} = w_n h_n + a_n(1) + \cdots + a_n(w_n). \tag{1.1}$$

We require that $w_n \geq 2$ for all n. Also, we require that the total measure be finite, i.e. $\sum_{n=1}^{\infty} |H_n \backslash H_{n-1}| < \infty$, so that T is defined on a finite measure space.

By rescaling by an appropriate constant r, we may assume $\cup_{n=0}^{\infty} H_n = [0,1] := X$. Let B_n denote the base of the tower H_n. Then

$$B_0 = [0, 1/r], \qquad B_n = [0, \frac{1}{r\, w_0 \ldots w_{n-1}}].$$

Thus, we consider T to be defined on $X = [0,1]$, endowed with the Borel sigma algebra and Lebesgue measure. By construction, T is a measure preserving invertible point transformation.

Put $f_n(x) = \frac{1}{\sqrt{|B_n|}} 1_{B_n}(x)$ the characteristic function of the n^{th}–base, normalized so that the 2–norm equals 1. Denote by $U_T f$ the operator $U_T f(x) = f(T^{-1}x)$. By construction of T, U_T is a unitary operator in $L^2(X)$.

Notice that

$$\mathcal{C} = \{\{T^k(B_n)\}_{k=0}^{h_n-1}\}_{n=0}^{\infty} \tag{1.2}$$

generates a dense subalgebra of the Borel σ–algebra, (here we are using the metric (modulo sets of measure zero) given by $d(A,B)$ =Lebesgue measure of $A \triangle B$). Then the subspace generated by the span of $\{U_T^k(f_n) : 1 \leq n < \infty, 0 \leq k < h_n\} =$ span of $\{1_{T^k(B_n)} : 1 \leq n < \infty, 0 \leq k < h_n\}$ is dense in $L^2(X)$.

§1.1. Spectral measures.

Given $T : X \mapsto X$ a measure preserving invertible transformation, to any $f \in L^2(X)$ there corresponds a positive measure σ_f on S^1, the unit circle, defined by $\hat{\sigma}_f(n) = \langle U_T^n f, f \rangle$. With the above notation, let $\sigma_n = \sigma_{f_n}$.

Definition 1.1. The maximal spectral type of T is the equivalence class of Borel measures σ on S^1 (under the equivalence relation $\mu_1 = \mu_2$ if and only if $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$), such that $\sigma_f << \sigma$ for all $f \in L^2(X)$ and if ν is another measure for which $\sigma_f << \nu$ for all $f \in L^2(X)$ then $\sigma << \nu$.

By the canonical decomposition of $L^2(X)$ into decreasing cycles (see appendix in Parry [10]) with respect to the operator U_T, there exists a Borel measure $\sigma = \sigma_f$ for some $f \in L^2(X)$, such that σ is in the equivalence class defining the maximal spectral type of T. By abuse of notation, we will call this measure the maximal spectral type measure, but it can be replaced by any other measure in its equivalence class.

Lemma 1.2. *σ is absolutely continuous with respect to $\sum_{n=0}^{\infty} 2^{-n}\sigma_n$.*

`Proof:` Given $f \in L^2(X)$, since the family $\mathcal{C}$ defined in (1.2) generates a dense subalgebra, f can be approximated by functions $g_n \in L^2(X)$ which are constant on the levels of the tower H_n. Then, $g_n = F_n(U_T)f_n$ for some polynomial $F_n(z)$, and

$$d\sigma_f = d\sigma_{g_n} + d\nu_n = |F_n|^2 d\sigma_n + d\nu_n$$

where $\|\nu_n\| \to 0$ as $n \to \infty$. Thus, if A is a set such that $\sigma_n(A) = 0$ for all n, then $0 \le \sigma_f(A) = \nu_n(A) \le \|\nu_n\| \to 0$ as $n \to \infty$. ∎

We will exploit a recurrence relationship between the spectral measures σ_n.

The bases B_n's are recursively related in the following fashion (see Figure 2):

$$B_n = B_{n+1} \cup T^{h_n+s_n(1)}B_{n+1} \cup T^{2h_n+s_n(2)}B_{n+1} \cup \ldots \cup T^{(w_n-1)h_n+s_n(w_n-1)}B_{n+1},$$
$$|B_n| = w_n|B_{n+1}|$$

where $s_n(k) = a_n(1) + \cdots + a_n(k)$. Letting

$$P_n(z) = \frac{1}{\sqrt{w_n}} \sum_{k=0}^{w_n-1} z^{kh_n+s_n(k)} \tag{1.3}$$

where $s_n(0) = 0$, we obtain $f_n = P_n(U_T)f_{n+1}$. Iterating this relationship, we have

$$d\sigma_n = |P_n|^2 d\sigma_{n+1} = \cdots = \prod_{j=0}^{m-1} |P_{n+j}|^2 d\sigma_{n+m}. \tag{1.4}$$

Thus, σ_n is absolutely continuous with respect to σ_{n+m} for all $m \ge 0$, and the continuous parts of these two measures are equivalent.

Theorem 1.3. *Let $d\rho_n = \prod_{j=0}^{n} |P_j|^2 d\lambda$. Then $\hat{\rho_n}(k) - \hat{\sigma_0}(k) \to 0$ as $n \to \infty$. In other words, (since $\|\rho_n\| = 1$) σ_0 is the weak$*$ limit of the ρ_n. (λ denotes the normalized Lebesgue measure).*

`Proof:` A proof of this theorem can be found in Choksi & Nadkarni [5]. Another proof which was kindly supplied by a referee is the following:

Let $R_n = P_0 \ldots P_n$ and $Q_n = |R_n|^2$. To show that $d\sigma_0 = w^* \lim_{n\to\infty} Q_n d\lambda$ it suffices to show that $\hat{\sigma}_0(m) - \hat{Q}_n(m) \to 0$ as $n \to \infty$ for all m. But $d\sigma_0 = Q_n d\sigma_{n+1}$, and

$$|\hat{\sigma}_0(m) - \hat{Q}_n(m)| = |\sum_{j\ne 0} \hat{Q}_n(m-j)\hat{\sigma}_{n+1}(j)| \le \sum_{|j|\ge h_{n+1}} |\hat{Q}_n(m-j)| \tag{1.5}$$

because $|\hat{\sigma}_{n+1}(j)| \le 1$ for all j and $|\hat{\sigma}_{n+1}(j)| = 0$ for $0 < |j| < h_{n+1}$ by definition of σ_{n+1}. On the other hand, by (1.1) and (1.3), $\deg(P_n) = h_{n+1} - h_n - a_n(w_n)$ (see definition of deg in (2.2)). Then by construction,

$$\deg(R_n) = \deg(P_0) + \cdots + \deg(P_n) \le \sum_{k=0}^{n} h_{k+1} - h_k = h_{n+1} - h_0 < h_{n+1}.$$

Also, $\deg(Q_n) = \deg(R_n)$, and hence $\hat{Q}_n(k) = 0$ for $|k| \ge h_{n+1}$. Moreover, $\hat{R}_n(k) = 0$ or $(w_0 \ldots w_n)^{-1/2}$ for all k by definition of R_n. Since $Q_n = R_n\bar{R}_n$, we have that for $|k| < h_{n+1}$,

$$0 \le \hat{Q}_n(k) \le \frac{h_{n+1} - |k|}{w_0 \ldots w_n}. \tag{1.6}$$

From (1.5) and (1.6) it follows that if $|m| < h_{n+1}$,

$$|\hat{\sigma}_0(m) - \hat{Q}_n(m)| \leq 2\frac{m^2}{w_0 \dots w_n} \to 0 \quad \text{as} \quad n \to \infty,$$

finishing the proof. ■

The same argument shows that each of the measures σ_n enjoys this property, that is, they are the weak$*$ limit of the measures obtained by replacing $d\sigma_{n+m}$ in equation (1.4) by the Lebesgue measure.

We will show that σ_0 is singular to Lebesgue measure for a class of rank one transformations whose cutting numbers $\{w_n\}_{n=0}^{\infty}$ do not grow too rapidly.

Theorem 1.4. *If* $\sum_{n=1}^{\infty} (1/w_n)^2 = \infty$*, then* $\sigma_0 \perp \lambda$.

The proof of Theorem 1.4 also shows that σ_n is singular to Lebesgue measure for all $n \geq 0$. Then, by Lemma 1.2, σ is also singular to Lebesgue measure.

Corollary 1.5. *If* $\sum_{n=1}^{\infty} (1/w_n)^2 = \infty$*, then* $\sigma \perp \lambda$.

Properties implying the singularity of Riesz products have been studied for some time (see [7], [9], [11], and [12] for some references). In particular it is known that a classical Riesz product is singular if its coefficients are not in ℓ^2. In §2 we adapt the proof of this result given by Peyrière [11] to the generalized Riesz products ρ_n, and thus prove Theorem 1.4. Another proof can be obtained by adapting Bourgain's approach in [4]. In fact we shall borrow some ideas from both methods.

§1.2. Comments and remarks.

Certain classes of measure preserving transformations are already known to have singular spectral type. For example, Baxter [3] proved that any α–rigid transformation, with $\alpha > 1/2$, has singular spectral type. (An α–rigid transformation is a measure preserving transformation such that there exists a sequence $\{n_k\}_{k=1}^{\infty}$ for which $\lim_{k\to\infty} m(T^{n_k}A \cap A) \geq \alpha m(A)$ for all measurable sets A.)

Naturally, one asks the following question.

Question 1.6. *Does any α–rigid transformation have singular spectral type?*

It follows from Theorem 1.4, that for any $0 < \alpha < 1$, one can construct α–rigid (but not β–rigid for $\beta > \alpha$) rank one transformations with singular spectral type. See Example 3.1 bellow. Moreover, one can construct rank one transformations without rigidity which enjoy this property:

Proposition 1.7. *There are mixing rank one transformations with singular spectral type.*

Bourgain's result [4] on the spectral type of Ornstein's mixing rank one transformation already proves this proposition. However, Ornstein's transformation involves a random construction. In §3, we give a proof of Proposition 1.7 using an explicit construction.

These examples seem to support the suspicion of many that singular spectral type may be a characteristic of rank one transformations.

Conjecture 1.8. *Every rank one transformation has singular spectral type.*

A positive answer to this conjecture would be the link between Kalikow's and Host's results on the problem of whether 2–fold mixing implies 3–fold mixing, since the first proved it for mixing rank one transformations and the second for transformations with singular spectral type.

Lastly, we mention that the condition in Theorem 1.4 is, of course, not the best possible. Indeed, we can construct transformations violating the hypothesis of the theorem but which have singular spectral type. The simplest example is the rank one transformation obtained by setting $w_n = n$ and $a_n(i) = 0$ for $1 \le i \le n$, for all n. Since no extra steps are added on any tower, this transformation is rigid (i.e. 1–rigid) and, by Baxter's condition, has singular spectral type.

A natural conjecture, intermediate between Theorem 1.4 and Conjecture 1.8, is the following:

Conjecture 1.9. *If $\sum_{n=1}^{\infty} c_n^2 = \infty$ for some choices of coefficients c_n of the polynomials $|P_n|^2$, then $\sigma \perp \lambda$.*

§2. Proof of Theorem 1.4.

Lemma 2.1. *Let $\{P_n\}_{n=1}^{\infty}$ be a family of trigonometric polynomials on S^1 with positive coefficients. Let $d\rho_n = \prod_{k=1}^{n} |P_k|^2 d\lambda$. If $\|P_n\|_2 = 1$ and $\|\prod_{i=1}^{n} P_i\|_2 = 1$ for all n, then $\lim_{n\to\infty} \rho_n$ exists in the weak$*$ topology.*

`Proof:` By the hypothesis, the measures $\{\rho_n\}_{n=1}^{\infty}$ satisfy:

(a) ρ_n is a probability measure on S^1 for all n,

(b) $\hat{\rho}_{n+1}(j) \ge \hat{\rho}_n(j)$ for all j and n.

Indeed, let $Q_n = \prod_{j=1}^{n} |P_j|^2$. By hypothesis, $|P_{n+1}|^2 = 1 + R_n$ where R_n is a trigonometric polynomial with positive coefficients and $\hat{R}_n(0) = 0$. Thus

$$\hat{\rho}_{n+1}(j) = \hat{Q}_n(j) + \hat{Q}_n * \hat{R}_n(j) \ge \hat{Q}_n(j) = \hat{\rho}_n(j).$$

From (a) and (b) it follows that $\lim_{n\to\infty} \rho_n$ exists in the weak$*$ topology. ■

This lemma implies that all of the weak$*$ limits we shall write down later actually exist. So from now on we omit mentioning this fact every time.

Proposition 2.2. *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of trigonometric polynomials as in Lemma 2.1. The following two conditions are equivalent:*

(a) $\inf\{\|P_{n_1} \dots P_{n_k}\|_1 ; n_1 < n_2 < \dots < n_k\} = 0$,

(b) *the measure $d\mu = w^* \lim_{n\to\infty} \prod_{j=1}^{n} |P_j|^2 d\lambda$ is singular.*

Notes:

1. This proposition is the key element in our proof. It allows us to drop to a subsequence, instead of working with the full sequence of polynomials:

Corollary 2.3. *Let $\{P_n\}_{n=1}^{\infty}$ and μ be as in Proposition 2.2. Let $\{P_{n_k}\}_{k=1}^{\infty}$ be a subsequence and let $d\nu = w^* \lim_{k\to\infty} \prod_{j=1}^{k} |P_{n_j}|^2 d\lambda$. If ν is singular, so is μ.*

2. The direction (a) $\Rightarrow$ (b) is also a key element in Bourgain's proof in [4]. We are using this idea as well as the implication (b) $\Rightarrow$ (a).

The proof of this proposition relies heavily on the fact that the measure is the weak$*$ limit of such products. One can easily construct a non–singular measure which is a weak$*$ limit of functions $|f_n|^2$ with $\|f_n\|_2 = 1$ and $\int |f_n(x)|dx \to 0$ as $n \to \infty$.

`Proof of Proposition 2.2`: Let $\lambda =$ Lebesgue measure, and $d\mu = w^* \lim_{N\to\infty} \prod_{n=1}^{N} |P_n|^2 d\lambda$. By Lemma 2.1, the limit measure μ exists and is a probability measure. Also, for any fixed finite sequence $n_1 < \ldots < n_k$, the measure $d\alpha = w^* \lim_{N\to\infty} \prod_{\substack{n=1 \\ n\neq n_1,\ldots,n_k}}^{N} |P_n|^2 d\lambda$ exists and is a probability measure. Moreover, $d\mu = f^2 d\alpha$, where $f = |P_{n_1} \cdots P_{n_k}|$.

(a) $\Rightarrow$ (b): To prove that $\mu \perp \lambda$, it suffices to show that for any $\epsilon > 0$, there is a set E with $\lambda(E) < \epsilon$ and $\mu(E^c) < \epsilon$. Let $0 < \epsilon < 1$.

Choose $n_1 < \ldots < n_k$ such that $\int f d\lambda < \epsilon^2$. By Chebyshev's inequality, the set $E = \{f > \epsilon\}$ satisfies:

$$\lambda(E) \le \|f\|_1/\epsilon \le \epsilon^2/\epsilon = \epsilon,$$

and

$$\mu(E^c) = \int_{E^c} d\mu = \int_{E^c} f^2 d\alpha \le \int_{E^c} \epsilon^2 d\alpha \le \epsilon^2 < \epsilon.$$

(b)$\Rightarrow$(a): Given $0 < \epsilon < 1$, there exists a continuous function φ such that:

$$0 \le \varphi \le 1, \qquad \mu(\{\varphi \ne 0\}) < \epsilon, \qquad \text{and} \quad \lambda(\{\varphi \ne 1\}) < \epsilon.$$

Let $f_N = \prod_{n=1}^{N} |P_n|$. Let $A = \{\varphi \ne 1\}$, then

$$\begin{aligned}\int f_N \; d\lambda &= \int_A f_N \; d\lambda + \int_{A^c} f_N \; d\lambda \\ &\le \lambda(A)^{1/2} + (\int_{A^c} f_N^2 \; d\lambda)^{1/2} \lambda(A^c)^{1/2} \\ &\le \sqrt{\epsilon} + (\int f_N^2 \, \varphi \; d\lambda)^{1/2}.\end{aligned}$$

But since $d\mu = w^* \lim_{N\to\infty} |f_N|^2 d\lambda$,

$$\lim_{N\to\infty} \int f_N^2 \, \varphi \; d\lambda = \int \varphi \; d\mu \le \mu(\{\varphi \ne 0\}) < \epsilon.$$

Thus, taking N sufficiently large, $\int f_N \; d\lambda < 2\sqrt{\epsilon}$. Since ϵ is arbitrary, $\lim_{N\to\infty} \int f_N \; d\lambda = 0$. ∎

The next theorem illustrates Peyrière's technique and is the backbone of the proof of Theorem 1.4.

Theorem 2.4. *(Peyrière [11]) Let μ be a Borel probability measure on S^1. If there exists an increasing sequence of integers $\{m_k\}_{k=1}^{\infty}$ such that*
(a) $\hat{\mu}(m_k) = a_k$, *and* $\{a_k\}_{k=1}^{\infty} \notin \ell^2$ *and*
(b) $\hat{\mu}(m_k - m_j) = a_k \bar{a}_j$ *if* $k \neq j$,
then $\mu \perp \lambda$.

Proof: Let $f_k(z) = z^{m_k}$. Then $\{f_k\}_{k=1}^{\infty}$ is a bounded orthogonal system in $L^2(\lambda)$, and $\{(f_k - \bar{a}_k)\}_{k=1}^{\infty}$ is an orthogonal system in $L^2(\mu)$. Also, since μ is a finite measure, $\|f_k - \bar{a}_k\|_{L^2(\mu)}$ is bounded in k.

Let $\{c_k\}_{k=1}^{\infty} \in \ell^2$ be a sequence such that $\bar{a}_k c_k \geq 0$ and $\sum_{k=1}^{\infty} \bar{a}_k c_k = \infty$. Such a sequence exists since $\{a_k\}_{k=1}^{\infty} \notin \ell^2$. Then, the sequences of functions $\sum_{k=1}^{n} c_k f_k$ and $\sum_{k=1}^{n} c_k (f_k - \bar{a}_k)$ converge in $L^2(\lambda)$ and $L^2(\mu)$ respectively. Thus, there is a sequence n_j such that

$$\sum_{k=1}^{n_j} c_k f_k \quad \text{converges } \lambda\text{–a.e. as } j \to \infty,$$

and

$$\sum_{k=1}^{n_j} c_k (f_k - \bar{a}_k) \quad \text{converges } \mu\text{–a.e. as } j \to \infty.$$

Then, both series cannot converge for the same z because their difference is

$$\sum_{k=1}^{n_j} c_k \bar{a}_k \underset{j}{\longrightarrow} \infty.$$

Hence, the set E on which the first series converges is a Borel set such that $\lambda(E^c) = 0$ and $\mu(E) = 0$, which ends the proof. ∎

§2.2. Fourier coefficients of generalized Riesz products.

In light of Theorem 2.4, we need to look at the Fourier coefficients of the generalized Riesz products defining σ_0.

Recall, from equation (1.3), that the polynomial P_n has the form

$$P_n(z) = \frac{1}{\sqrt{w_n}}(z^{c_0} + z^{c_0+c_1} + z^{c_0+c_1+c_2} + \cdots + z^{c_0+c_1+c_2+\cdots+c_{w_n-1}}),$$

where $c_0 = 0$, $c_i = h_n + a_n(i)$, $i = 1, \ldots, w_n - 1$. That is, $c_1, c_2, \ldots, c_{w_n-1}$ are the heights of the first $w_n - 1$ subcolumns of the tower H_n. Now form the product

$$R_n = P_n \bar{P}_n = 1 + \frac{1}{w_n} \sum_{0 \leq i \neq j \leq w_n - 1} z^{(c_0+\cdots+c_i)-(c_0+\cdots+c_j)}. \tag{2.1}$$

Define the degree of any trigonometric polynomial f by

$$\deg(f) = \max\{|k| : \hat{f}(k) \neq 0\} \qquad (2.2).$$

Let $d_n = \deg(P_n) = \deg(R_n)$. From equations (1.1) and (1.3) of the introduction, we have

$$d_n = h_{n+1} - h_n - a_n(w_n) < h_{n+1}, \tag{2.3}$$

$$h_n \le h_{n+1}/w_n \le h_{n+1}/2. \tag{2.4}$$

With the help of Proposition 2.2, instead of working with the full sequence of polynomials $\{P_n\}_{n=0}^{\infty}$, we can drop to a subsequence. Indeed, to show that σ_0 is a singular measure, it suffices to show that $\mathrm{w}^* \lim_{k\to\infty} \prod_{j=1}^{k} R_{n_j}\, d\lambda$ is a singular measure for some sequence $\{n_k\}_{k=1}^{\infty}$.

Assume $\{n_k\}_{k=1}^{\infty}$ is a sequence satisfying $n_{k+1} \ge n_k + 3$. Let $Q_k = R_{n_1} \ldots R_{n_k}$. Then, from (2.3) and (2.4) it follows that

$$h_{n_k+1} \le \frac{1}{4} h_{n_{k+1}}, \tag{2.5}$$

and telescoping, since $n_j + 1 \le n_{j+1}$,

$$\begin{aligned} q_k :=& \deg(Q_k) = d_{n_1} + d_{n_2} + \cdots + d_{n_k} \\ \le& (h_{n_1+1} - h_{n_1}) + (h_{n_2+1} - h_{n_2}) + \cdots + (h_{n_k+1} - h_{n_k}) \\ <& h_{n_k+1}. \end{aligned} \tag{2.6}$$

Now we will look at the Fourier coefficients of the polynomials Q_k. First, we need to examine R_{n_k}.

Under the above hypothesis, the polynomial R_{n_k} has isolated Fourier coefficients at 0 and d_{n_k}, in the sense that $\hat{R}_{n_k} = 0$ on a large interval of integers around 0 and d_{n_k}.

Lemma 2.5. *If $\{n_k\}_{k=1}^{\infty}$ is a sequence such that $n_{k+1} \ge n_k + 3$, then the Fourier coefficients of R_{n_k} satisfy:*

(a) $\hat{R}_{n_k}(0) = 1,$ *and* $\hat{R}_{n_k}(n) = 0, \quad 0 < |n| < h_{n_k},$

(b) $\hat{R}_{n_k}(d_{n_k}) = 1/w_{n_k}$ *and* $\hat{R}_{n_k}(n) = 0, \quad d_{n_k} - h_{n_k} < |n| < d_{n_k}.$

Proof: Recalling that R_{n_k} is given by (2.1), let $d_{i,j} = (c_0 + \cdots + c_i) - (c_0 + \cdots + c_j)$, $i \ne j$. Since $d_{i,j} \ne 0$ if $i \ne j$, it is immediate that $\hat{R}_{n_k}(0) = 1$. The rest of (a) follows from the fact that for $i > j$, $d_{i,j} \ge c_i \ge h_{n_k}$, and by symmetry, $|d_{i,j}| \ge h_{n_k}$ for $i < j$ also.

To prove (b), note that $d_{n_k} = \max d_{i,j} = d_{(w_{n_k}-1),0}$. This implies our claim that $\hat{R}_{n_k}(d_{n_k}) = 1/w_{n_k}$.

Lastly, suppose $i > j$ and (i, j) is not the pair $(w_{n_k} - 1, 0)$. Then $d_{i,j} = c_{j+1} + \cdots + c_i$ is a sum over a proper subset of the indexes $\{1, 2, \ldots, w_{n_k} - 1\}$. Therefore, $d_{n_k} - d_{i,j}$, being the sum over the complement of this subset, satisfies $d_{n_k} - d_{i,j} \ge \min\{c_1, \ldots, c_{w_{n_k}-1}\} \ge h_{n_k}$. That is, there is a gap of at least h_{n_k} between d_{n_k} and the previous non–zero Fourier coefficient, which proves (b). ■

Lemma 2.6. *With the above notation, if $\{n_k\}_{k=1}^{\infty}$ is a sequence such that $n_{k+1} \ge n_k + 3$, then the Fourier coefficients of the $\{Q_k\}_{k=1}^{\infty}$ satisfy:*

(a) $\hat{Q}_{k+m}(n) = \hat{Q}_k(n)$ *whenever* $|n| \le q_k, \quad m \ge 0,$

(b) $\hat{Q}_k(0) = 1$ *and* $\hat{Q}_k(d_{n_k}) = \dfrac{1}{w_{n_k}}.$

`Proof:` Property (a) and the fact that $\hat{Q}_k(0) = 1$ are immediate consequences of (2.5), (2.6) and part (a) of Lemma 2.5.

Now consider the coefficients of $Q_{k+1} = Q_k R_{n_{k+1}}$ on the interval $[d_{n_{k+1}} - q_k, d_{n_{k+1}} + q_k]$. Since $q_k < h_{n_{k+1}}/4$ (see (2.5) and (2.6)), it is clear that (using Lemma 2.5 (b) for $R_{n_{k+1}}$):

$$\hat{Q}_{k+1}(n + d_{n_{k+1}}) = \hat{Q}_k(n)\frac{1}{w_{n_{k+1}}} \quad \text{for all } n \in [-q_k, q_k]. \tag{2.7}$$

In particular, $\hat{Q}_{k+1}(d_{n_{k+1}}) = \hat{Q}_k(0)/w_{n_{k+1}} = 1/w_{n_{k+1}}$ which proves (b). ∎

Given a sequence $n_1 < n_2 < \ldots$, define α to be the probability measure $d\alpha = w^* \lim_{k\to\infty} \prod_{j=1}^{k} |P_{n_j}|^2 d\lambda$.

Lemma 2.7. *Let $\{n_j\}_{j=1}^{\infty}$ be a sequence satisfying $n_{j+1} \geq n_j + 3$. Then there is a sequence $\{m_j\}_{j=1}^{\infty} \subset \mathbf{N}$ such that:*

(a) $$\hat{\alpha}(\pm m_j) = 1/w_{n_j},$$

(b) $$\hat{\alpha}(m_j \pm m_k) = \hat{\alpha}(m_j)\hat{\alpha}(m_k), \ j \neq k.$$

`Proof:` By definition of α and with the notation preceding the lemma, we have $\hat{\alpha}(n) = \lim\limits_{k\to\infty} \hat{Q}_k(n)$ for all n.

Let $m_k = d_{n_k}$. Then, from Lemma 2.6, it follows that

$$\hat{\alpha}(n) = \hat{Q}_k(n) \quad \text{whenever } |n| \leq q_k, \tag{2.8}$$

and that $\hat{\alpha}(m_k) = 1/w_{n_k}$ which proves (a), since α is real.

To prove (b), let $j < k$ and apply equation (2.7) for k instead of $k+1$, with $n = \pm m_j$. Noting that $m_j \in$ support of $\hat{Q}_j \subset$ support of $\hat{Q}_{k-1} \subset [-q_{k-1}, q_{k-1}]$, we get from equations (2.7) and (2.8) that

$$\begin{aligned}\hat{\alpha}(m_k \pm m_j) = \hat{Q}_k(m_k \pm m_j) &= \hat{Q}_{k-1}(\pm m_j)\frac{1}{w_{n_k}} \\ &= \hat{\alpha}(\pm m_j)\hat{\alpha}(m_k) = \hat{\alpha}(m_j)\hat{\alpha}(m_k),\end{aligned}$$

which proves (b). ∎

`Proof of Theorem 1.4:` We will apply Lemma 2.7 to a sequence $\{n_j\}_{j=1}^{\infty}$ which in addition satisfies $\sum_{j=1}^{\infty} 1/w_{n_j}^2 = \infty$.

By hypothesis, we have

$$\infty = \sum_{n=1}^{\infty} \frac{1}{w_n^2} = (\sum_{n=3j} + \sum_{n=3j-1} + \sum_{n=3j-2}).$$

So, at least one of the three sums is ∞. Thus we can choose $\{n_j\}_{j=1}^{\infty}$ such that $n_{j+1} = n_j + 3$ and $\sum_{j=1}^{\infty} 1/w_{n_j}^2 = \infty$. By Lemma 2.7, the measure $d\alpha = w^* \lim_{k\to\infty} \prod_{j=1}^{k} |P_{n_j}|^2 d\lambda$ satisfies the hypothesis of Theorem 2.4. Thus, $\alpha \perp \lambda$, and by Corollary 2.3, $\sigma_0 \perp \lambda$. ∎

§3. Examples.

Example 3.1. *Any rank one transformation with* $\liminf w_n < \infty$ *has singular spectral type.*

The proof is immediate from Theorem 1.4. This shows that, for example, Chacon's transformation with $w_n = 2$ and $a_n(1) = 0$, $a_n(2) = 1$, which is 1/2–rigid, has singular spectral type. Similarly, for arbitrarily small $\alpha > 0$ one can construct α–rigid (but not β–rigid for $\beta > \alpha$) rank one transformations with singular spectral type. That is, for $\alpha = 1/M$, let $w_n = M$, $a_n(k) = k - 1$ for $k = 1, \dots, M-1$ and $a_n(M) = 0$.

Next, to prove Proposition 1.7, we recall a construction of Adams and Friedman [2].

Definition 3.2 A *staircase construction* is a rank one transformation T such that, at each step, the numbers of intervals added, $a_n(k)$, are defined by

$$a_n(1) = a_n(w_n) = 0, \qquad a_n(k) = k - 1 \quad \text{for} \quad 2 \le k \le w_n - 1.$$

Thus, every staircase construction is completely determined by the sequence $\{w_n\}_{n=1}^{\infty}$. We denote such a transformation by $T = T_{\{w_n\}}$.

Let $\{r_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be two sequences defined inductively as follows. $m_1 = 1$ and $r_1 \ge 3$. Assume that $m_1, \dots, m_{k-1}$ and $r_1, \dots, r_{k-1}$ have already been defined. Let $s_j = r_l$ for $m_l \le j < m_{l+1}$, $l < k-1$, and $s_j = r_{k-1}$ for $j \ge m_{k-1}$. The transformation $S = T_{\{s_j\}}$ is "uniform Çesaro" (see [2]). In particular, there exists N such that for $n \ge N$

$$\|\frac{1}{n}\sum_{i=0}^{n-1} S^{il}\chi_I(x) - \lambda(I)\|_1 \le \frac{1}{k}\lambda(I)$$

for all positive integers l and all levels I in the m_{k-1} tower. Choose r_k such that

$$r_k \ge kN$$

and choose m_k such that the height h_{m_k} of the m_k tower satisfies

$$h_{m_k} \ge k r_k^2.$$

With the sequences $\{r_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ as above, construct $\{w_n\}_{n=1}^{\infty}$ by putting

$$w_n = r_k \quad \text{for} \quad m_k \le n < m_{k+1}.$$

Theorem 3.3. *(Adams & Friedman [2]) The staircase* $T = T_{\{w_n\}}$ *with* $\{w_n\}_{n=1}^{\infty}$ *as above, is mixing.*

`Proof of Proposition 1.7`: To define the example, choose a staircase transformation as in Theorem 3.3, with the additional property that $m_{k+1} - m_k \ge r_k^2$. Then

$$\sum_{n=1}^{\infty} \frac{1}{w_n^2} = \sum_{k=1}^{\infty} \sum_{n=m_k}^{m_{k+1}-1} \frac{1}{r_k^2} \ge \sum_{k=1}^{\infty} 1 = \infty,$$

and we can apply Theorem 1.4. ■

Another example which proves Proposition 1.7 is the standard staircase transformation ($w_n = n$). This transformation has recently been shown to be mixing by Adams [1], and also to have singular spectral type by Klemes [8].

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