ON APPROACH REGIONS FOR THE CONJUGATE POISSON INTEGRAL AND SINGULAR INTEGRALS.

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Abstract

Let $\tilde{u}$ denote the conjugate Poisson integral of a function $f \in L^p(R)$. We give conditions on a region Ω so that $\lim_{\substack{(v,\epsilon)\to(0,0)\\(v,\epsilon)\in\Omega}} \tilde{u}(x+v,\epsilon) = Hf(x)$, the Hilbert transform of f at x, for a.e. x. We also consider more general Calderón–Zygmund singular integrals and give conditions on a set Ω so that $\sup_{(v,r)\in\Omega} |\int_{|t|>r} k(x+v-t)f(t)dt|$ is a bounded operator on L^p, $1 < p < \infty$, and is weak (1,1).

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Let $f \in L^p(R^d)$ and let $u(x,y)$ denote the Poisson integral of f. Then a classical theorem of Fatou [3] asserts that u has non–tangential limits a.e. on R^d. In 1984, Nagel and Stein [5] considered more general convergence than the classical non–tangential convergence and gave necessary and sufficient conditions for an approach region Ω so that convergence occurs if $u(x,y)$ approaches the boundary through the region Ω.

In this paper we consider the associated problem for the conjugate Poisson integral of a function f, as well as for more general Calderon–Zygmund singular integrals.

Let $k(x)$ be a Calderón–Zygmund kernel on R^d, that is, $k(x) = \frac{w(x)}{|x|^d}$, where:

(k1) w is homogeneous of degree 0 and $w \in L^\infty(S^{d-1})$,
(k2) its integral over the S^{d-1} sphere vanishes, and
(k3) $|k(x+y) - k(x)| \leq C|y|/|x|^{d+1}$, if $|x| > 2|y|$.

Let $k_1(x) = k(x)$ if $|x| > 1$ and 0 otherwise, and define $k_r(x) = r^{-d} k_1(x/r)$. Consider the d–dimensional singular integral defined by this kernel,

$$H_r f(x) = \int_{|x-t|>r} f(t) k(x-t) dt = f * k_r(x).$$

Given a set $\Omega \subset R^d \times R^+$, consider the maximal transform

$$H_\Omega^\# f(x) = \sup_{(v,r)\in\Omega} |H_r f(x+v)|.$$

We will also use the notation

$$H^\# f(x) = \sup_{r>0} |H_r f(x)|, \qquad Hf(x) = \lim_{r\to 0} H_r f(x),$$

and the standard Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x+t)| dt.$$

In this paper we find necessary and sufficient conditions on the sets Ω for which $H_\Omega^\# f$ is a weak (1,1) and strong (p,p) operator, $1 < p < \infty$. It turns out that such sets coincide with those Ω's for which the moved Hardy–Littlewood's maximal operator

$$M_\Omega f(x) = \sup_{(v,r)\in\Omega} \frac{1}{|B(v,r)|} \int_{B(v,r)} |f(x+t)| dt$$

is a weak (1,1) and strong (p,p) operator, $1 < p < \infty$. Nagel and Stein [5] showed that a necessary and sufficient condition for $M_\Omega f$ to be weak (1,1) and strong (p,p), $1 < p < \infty$, is that the set Ω satisfies the following condition, known as the cone condition.

Definition 1. *We say that a set* $\Omega \subset R^d \times R^+$ *satisfies the* **cone condition** *if for any* α, *the set*

$$\Omega_\alpha = \{(x,y) \in R^d \times R^+ : \exists (v,r) \in \Omega \text{ such that } |x-v| < \alpha(y-r)\}$$

has the property that there exists a constant $C = C(\alpha)$ *such that the cross section set*

$$\Omega_\alpha(\lambda) = \{x \in R^d : (x,\lambda) \in \Omega_\alpha\}$$

satisfies

$$|\Omega_\alpha(\lambda)| \leq C\lambda^d,$$

for all $\lambda > 0$.

In the first section we show that if Ω satisfies the cone condition then $H_\Omega^\# f$ and $\sup_{(v,r)\in\Omega} |Q_r * f(x+v)|$, the maximal function associated with conjugate Poisson kernel, are weak (1,1) and strong (p,p) operators, $1 < p < \infty$. The sufficiency of the cone condition in the one dimensional case was already proved in S. Ferrando's Ph.D. thesis [4]. Ferrando reduced the problem to the case in which Ω is a discrete set and proved the result using a covering argument plus a discrete version of the Hilbert transform. In this work, we extend the result to R^d by using an argument involving atomic decompositions for functions in R_+^{d+1}.

In section 2, we show that the cone condition is also necessary for $H_\Omega^\#$ to be weak (p,p), $1 \leq p < \infty$, when $k(x)$ is any of the Riesz kernels. In section 3, we show the existence of the limit of $H_r f(x+v)$ as (v,r) approach $(0,0)$ on a region satisfying the cone condition. We apply this result to the convergence of $Q_y f(x+v)$, the conjugate Poisson integral of f, when (v,y) tends to (0,0) on an approach region Ω satisfying the cone condition. Lastly, in section 4, we apply the results to the Ergodic theory setting.

§1. Maximal estimates.

The proof that the maximal operator $H_\Omega^\#$ is weak (1,1) and strong (p,p) for $1 < p < \infty$, will make use of the atomic decomposition for operators in R_+^{d+1}. This approach was suggested to us by E. M. Stein, greatly simplifying our original proof.

The atomic decomposition allows us to reduce the problem of showing that $H_\Omega^\# f$ is a weak (1,1) and strong (p,p), $1 < p < \infty$, to showing that a simpler operator is of the same type.

Let $\tilde{\Omega} = \{(x,y) : (x,y_0) \in \Omega \text{ for some } y_0 \leq y\}$. Then $H_{\tilde{\Omega}}^\# f(x) \geq H_\Omega^\# f(x)$, and if Ω satisfies the cone condition, so does $\tilde{\Omega}$ because $\tilde{\Omega}_\alpha = \Omega_\alpha$ (see Definition 1). Therefore, there is no harm in working with the extended set $\tilde{\Omega}$ instead, which simplifies the proof.

Let $\Gamma = \{(v,t) \in R_+^{d+1} : |v| < t\}$. That is, Γ is a single cone positioned at $(0,0)$. Then $H_\Gamma^\# f(x) = \sup_{(v,r)\in\Gamma} |H_r f(x+v)|$ is the standard nontangential maximal function for the associated singular integral operator.

Theorem 2. *If* Ω *satisfies the cone condition, then*
a) $\int_{R^d} |H_{\tilde{\Omega}}^\# f(x)|^p\,dx < c_p \int_{R^d} |H_\Gamma^\# f(x)|^p\,dx$, *for* $0 < p < \infty$,
b) $|\{x \in R^d | H_{\tilde{\Omega}}^\# f(x) > \lambda\}| \leq c |\{x \in R^d | H_\Gamma^\# f(x) > \lambda\}|$,

c) $H^{\#}_{\Gamma} f(x) \leq H^{\#} f(x) + C(d) Mf(x)$, *and*
d) $H^{\#}_{\tilde{\Omega}} f$ *is a weak (1,1) and strong (p,p) operator, for* $1 < p < \infty$.

`Proof:` Parts a) and b) are an application of the results contained in Stein's Harmonic Analysis [7], pages 68 and 69. For completeness, we include his argument.
a) An atom associated to a ball $B \subset R^d$, is a measurable function $a(x,t)$ supported in the tent $T(B) = \{(x,t) : |x| < r - t\} \subset R^{d+1}_+$, such that $\|a\|_\infty \leq 1/|B|$.

If $H^{\#}_{\Gamma} f(x) \in L^p(R^d)$ then we can apply the atomic decomposition to the function $|H_y f(x)|^p$. Hence, to prove a), it will be enough to consider the case where $p = 1$ and $H_y f(x) = a(x,y)$ is an atom. Further, by translation, we can assume that atom is supported in $T(B)$, where B is a ball of radius r centered at the origin.

By the properties of the atom a, we clearly have $\sup_{(v,y)\in\tilde{\Omega}} |a(x+v,y)| \leq 1/|B|$. If $\sup_{(v,y)\in\tilde{\Omega}} |a(x+v,y)| \neq 0$ then there is a $(v,y) \in \tilde{\Omega}$ such that $(x+v,y) \in T(B)$; that is, $|x+v| < r - y$. Since $(v,y) \in \tilde{\Omega}$, then $-x \in \tilde{\Omega}_1(r) = \Omega_1(r)$. Hence

$$|\{x| \sup_{(v,y)\in\Omega} |a(x+v,y)| \neq 0\}| \leq |\Omega_1(r)|,$$

and by assumption, $|\Omega_1(r)| \leq cr^d$. From this we get

$$\int_{R^d} \sup_{(v,y)\in\Omega} |a(x+v,y)| dx \leq \frac{1}{|B|} |\Omega_1(r)| \leq c. \tag{1}$$

Since (1) holds for atoms, a) holds in general (by Theorem 3.2.3 in [7]).

b) To prove b) we repeat the same proof, but replace the function $H_y f(x)$ by the characteristic function of the set where $|H_y f(x)| > \lambda$.

c) It is easy to see that the operator $H^{\#}_{\Gamma} f$ can be compared the maximal operator $H^{\#} f$. Indeed

$$|H_r f(x+v) - H_r f(x)| \leq \int_{|t|>2r} |f(x-t)||k(t-v) - k(t)| dt$$
$$+ \int_{\substack{|t-v|>r \\ |t|\leq 2r}} |f(x-t)|k(t-v)| dt + \int_{r<|t|\leq 2r} |f(x-t)|k(t)| dt.$$

By property (k1), $|k(x)| \leq c/|x|^d$, thus the last two terms are majorized by

$$c(d) \frac{1}{|B(0,2r)|} \int_{B(0,2r)} |f(x-t)| dt.$$

To handle the first term, recall that by (k3), $|k(t-v) - k(t)| \leq C|v|/|t|^{d+1}$, if $|t| > 2|v|$. Thus, if $|v| < r$,

$$|k(t-v) - k(t)| \leq C \frac{r}{|t|^{d+1}} = C\Phi_r(t), \qquad \text{for } |t| > 2r,$$

where $\Phi_r(t) = r^{-d}\Phi_1(t/r)$, and $\Phi_1(t) = |t|^{-d-1}$ for $|t| > 2$. Thus

$$\sup_{(v,r)\in\Gamma} |H_r f(x+v) - H_r f(x)| \leq C \sup_{r>0} |f| * \Phi_r(x) + c(d)\ Mf(x).$$

Since Φ_1 is an integrable function on R^d which radially decreases at infinity with an appropriate rate, $\sup_{r>0} |f| * \Phi_r(x)$ is also dominated by $Mf(x)$. Hence

$$\sup_{(v,r)\in\Gamma} |H_r f(x+v)| \leq \sup_{r>0} |H_r f(x)| + C(d)\ Mf(x),$$

finishing the proof of c).

d) The proof of d) is an straight forward application of a), b) and c). ■

Let $Q_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$ denote the conjugate Poisson kernel in R^2_+. For a set $\Omega \subset R^2_+$, let $Q^{\#}_{\Omega} f(x) = \sup_{(v,\epsilon)\in\Omega} |Q_\epsilon * f(x+v)|$. With this notation, the corresponding version of Theorem 2 also holds for this maximal operator.

Theorem 3. *If Ω satisfies the cone condition, then*
a) $\int_{R^d} |Q^{\#}_{\tilde{\Omega}} f(x)|^p\, dx < c_p \int_{R^d} |Q^{\#}_{\Gamma} f(x)|^p\, dx$, *for* $0 < p < \infty$,
b) $|\{x \in R^d | Q^{\#}_{\tilde{\Omega}} f(x) > \lambda\}| \leq c |\{x \in R^d | Q^{\#}_{\Gamma} f(x) > \lambda\}|$,
c) $Q^{\#}_{\Gamma} f(x) \leq \frac{1}{\pi}[H^{\#}_{\Gamma} f(x) + c(d) Mf(x)]$, *and*
d) $H^{\#}_{\tilde{\Omega}} f$ *is a weak (1,1) and strong (p,p) operator, for* $1 < p < \infty$.

`Proof:` The proof is exactly the same as the proof of Theorem 2. ■

§2. Necessity of the cone condition.

The Riesz kernels in R^d are defined by the jth coordinate in the following way:

$$k_j(x) = \frac{w_j(x)}{|x|^d}, \quad \text{where } w_j(x) = \frac{x_j}{|x_j|}.$$

Proposition 4. *Let k be a Riesz kernel in R^d. If $H^{\#}_{\Omega} f$ is weak (p,p) for some $1 \leq p < \infty$ then Ω satisfies the cone condition.*

`Proof:` Recall

$$\Omega_\alpha = \{(x,t) : \exists (v,r) \in \Omega : |x - v| < \alpha(t - r)\}.$$

Without loss of generality we can assume $k(x) = k_1(x)$. For a fixed α, we need to estimate the measure of $\Omega_\alpha(\lambda) = \{x : (x,\lambda) \in \Omega_\alpha\}$ for any $\lambda > 0$.

Let $b \geq 2\alpha\lambda$ to be determined and

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x_1 \leq b \text{ and } |x_i| \leq \alpha\lambda \text{ for all } 2 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in \Omega_\alpha(\lambda)$ and $(v, r) \in \Omega$ such that $|x - v| < \alpha(\lambda - r)$. Then

$$
\begin{aligned}
|H_r f(v-x)| &= \Big| \int_{|t|>r} f(t-(v-x)) \frac{w_1(t)}{|t|^d} dt \Big| \\
&= \int_{\substack{|t|>r, \\ |v_1-x_1|<t_1<b+(v_1-x_1), \\ |t_i-(v_i-x_i)|<\alpha\lambda, i\neq 1}} \frac{1}{|t|^d} dt
\end{aligned}
$$

by the symmetry of the kernel.

Case 1: $r < \alpha\lambda$.

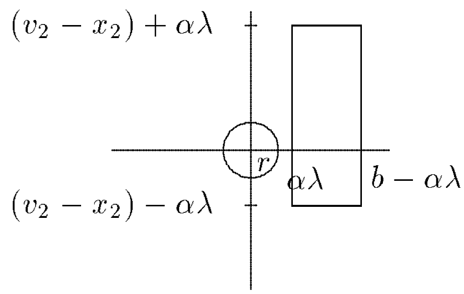

Figure 1

In this case, since $|x - v| < \alpha\lambda$,

$$
\begin{aligned}
H_r f(v-x)| &\geq \int_{\substack{\alpha\lambda<t_1<b-\alpha\lambda \\ |t_i-(v_i-x_i)|<\alpha\lambda, i\neq 1}} \frac{1}{|t|^d} dt \\
&\geq c(d) \frac{(b-2\alpha\lambda)(\alpha\lambda)^{d-1}}{(b+d\alpha\lambda)^d} \\
&= c(d) \frac{1}{(3+d)^d},
\end{aligned}
$$

if $b = 3\alpha\lambda$.

Case 2: $r \geq \alpha\lambda$

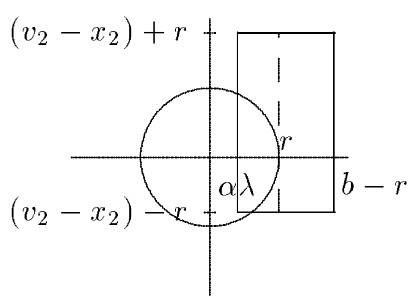

Figure 2

Now we will use also the fact that $|x - v| \leq \alpha(\lambda - r)$, so in particular, $r \leq \lambda$ and $\alpha \leq 1$.

$$
\begin{aligned}
H_r f(v-x)| &\ge \int_{\substack{r<t_1<b-\alpha\lambda \\ |t_i-(v_i-x_i)|<\alpha\lambda, i\neq 1}} \frac{1}{|t|^d} dt \\
&\ge c(d)\frac{(b-\alpha\lambda-r)(\alpha\lambda)^{d-1}}{(b+d\alpha\lambda)^d} \\
&\ge c(d)\frac{(b-2\lambda)(\alpha\lambda)^{d-1}}{(b+d\lambda)^d} \\
&= c(d)\alpha^{d-1}\frac{1}{(3+d)^d},
\end{aligned}
$$

if $b = 3\lambda$.

Let $A(\alpha) = c(d)/(3+d)^d$ if $\alpha \ge 1$, and $A(\alpha) = c(d)\alpha^{d-1}/(3+d)^d$ if $0 < \alpha < 1$. Then, if $H_\Omega^\# f$ is a weak (p,p) operator,

$$
\begin{aligned}
|\Omega_\alpha(\lambda)| &= |\{x : \exists (v,r) \in \Omega : |x-v| < \alpha(\lambda - r)\} \\
&\le |\{x : \sup_{\substack{(v,r)\in\Omega, \\ r\le\lambda}} H_r f(v-x) > A(\alpha)\}| \\
&\le |\{x : H_\Omega^\# f(-x) > A(\alpha)\}| \\
&\le \frac{C}{A(\alpha)^p}\|f\|_p^p = C(d,\alpha)\lambda^d.
\end{aligned}
$$

Hence Ω satisfies the cone condition. ■

§3. Almost everywhere convergence along Ω.

Let Ω satisfy the cone condition. In this section we prove pointwise convergence of $\lim_{\substack{(v,r)\to(0,0) \\ (v,r)\in\Omega}} H_r f(x+v)$ for any $f \in L^p(R^d)$, $1 \le p < \infty$.

Theorem 5. *Let $\Omega \subset R^d \times R^+$ satisfy the cone condition, such that $(0,0) \in \bar{\Omega}$. Then, for any $f \in L^p(R^d)$, $1 \le p < \infty$, $\lim_{\substack{(v,r)\to(0,0) \\ (v,r)\in\Omega}} H_r f(x+v) = Hf(x)$ a.e..*

`Proof:` Let $C_c^1(R^d)$ be the set of functions with compact support and continuous partial derivatives. Let $f \in C_c^1(R^d)$. Then

$$
\begin{aligned}
H_r f(x+v) &= f * k_1(x+v) + \int_{\{r<|x-y|<1\}} f(y+v)k(x-y)dy \\
&= I(x,v,r) + II(x,v,r).
\end{aligned}
$$

By continuity of f and compactness of its support, $I(x,v,r) \to f * k_1(x)$ as $(v,r) \to (0,0)$. For the second term, notice that by (k2), $\int_{\{r<|x-y|<1\}} k(x-y)dy = 0$, thus

$$
II(x,v,r) = \int [f(y+v) - f(x+v)]k(x-y)\chi_{\{r<|u|<1\}}(x-y)dy.
$$

Since the differential of f is continuous of compact support, the integrand is majorized by

$$c\,|x-y|^{-d+1}\ \chi_{\{0<|u|<1\}}(x-y)$$

which is integrable. And, as $(v,r)\to(0,0)$, the integrand converges to

$$[f(y)-f(x)]\ k(x-y)\ \chi_{\{0<|u|<1\}}(x-y).$$

From these two estimations,

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)=Hf(x) \qquad \text{for all} \quad x.$$

Now let $f\in L^p(R^d)$. Given $\epsilon>0$ choose $g\in C_c^1(R^d)$ such that $\|f-g\|_p<\epsilon$. Let

$$\Lambda f(x):=|\limsup_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)-\liminf_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)|.$$

Then, $\Lambda f=\Lambda(f-g)$ and, by Theorem 2,

$$|\{x:\Lambda f(x)>\alpha\}|=|\{x:\Lambda(f-g)(x)>\alpha\}|\le\frac{C(d)}{\alpha^p}\|f-g\|_p^p\le\frac{C(d)}{\alpha^p}\epsilon^p.$$

Since ϵ is arbitrary, the limit

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)$$

exists for almost every x.

Similar arguments show that $\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H_r f(x+v)=Hf(x)$ a.e.. ■

Theorem 6. *Let $Q_y(x)=\frac{1}{\pi}\frac{x}{x^2+y^2}$ denote the conjugate Poisson kernel in R_+^2. If Ω satisfies the cone condition, then*

$$\lim_{\substack{(v,\epsilon)\to(0,0)\\(v,\epsilon)\in\Omega}} Q_\epsilon * f(x+v) \quad \textit{exists for a.e. } x,$$

and is equal to $Hf(x)$.

`Proof:` It follows from Theorem 3 and the fact that

$$\lim_{\epsilon\to0} Q_\epsilon * f(x)=Hf(x),$$

(by arguments similar to those in Theorem 5). ■

§4. Hilbert transform for measurable flows.

Let (X, β, m) be a σ–finite measure space and $\{\tau_t\}_{t\in R^d}$ a measure preserving action of R^d acting on X, which is jointly measurable from $R^d \times X$ to X. We now will consider the truncated ergodic singular integrals

$$H'_r f(x) = \int_{r<|t|<1/r} f(\tau_t x) k(t) dt, \quad f \in L^p(X)$$

and the related moving maximal operator

$$H'^{\#}_{\Omega} = \sup_{(v,r)\in\Omega} |H'_r f(\tau_v x)|.$$

The singular integral results obtained in section 1 can be translated to this setting by means of a Calderón transfer principle. However, we first need to establish a modified version of the results in section 1, for the truncated singular integrals.

Since we are interested in the limit when $(v,r) \to (0,0)$, in this section we will assume that for all $(v,r) \in \Omega$, we have $r \leq 1$.

Corollary 7. *Let $\Omega \subset R \times R^+$ satisfy the cone condition. Then*

$$\sup_{(v,r)\in\Omega} \left| \int_{r<|t|<1/r} f(x+v+t) k(t) dt \right|$$

is a weak (1,1) and a strong (p,p) operator for $1 < p < \infty$.

`Proof:` The result follows from Theorem 2 because

$$\left| \int_{r<|t|<1/r} f(x+v+t) k(t) dt \right| \leq |H_r f(x+v)| + |H_{1/r} f(x+v)|,$$

and $\{(v, 1/r) : (v,r) \in \Omega\}$ satisfies the cone condition if $r \leq 1$. ■

Proposition 8. *(Transfer principle) Let $\Omega \subset R^d \times R^+$ and $1 \leq p < \infty$. If*

$$\sup_{(v,r)\in\Omega} \left| \int_{r<|t|<1/r} \varphi(x+v+t) k(t) dt \right|$$

is a weak (p,p) operator in $L^p(R)$, then $H'^{\#}_{\Omega} f$ is a weak (p,p) operator in $L^p(X)$.

`Proof:` Fix $M > 0$ and let $N = 3M$. Given $f \in L^p(X)$ define

$$\varphi_x(t) = \begin{cases} f(\tau_t x) & \text{if } |t| \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Then, for almost every x, $\varphi_x \in L^p(R^d)$. Indeed

$$\int_X \int_{R^d} |\varphi_x(t)|^p dt dx = \int_{|t|\leq N} \int_X |f(\tau_t x)|^p dx dt = c(d) N^d \|f\|_p^p,$$

because the flow is measure preserving.

Let $\Omega_M = \{(v,r) \in \Omega : |v| \le M, 1/M \le r \le M\}$. Then

$$\int_X |\{|s| \le M : \sup_{(v,r)\in\Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s+v-t)dt| \ge \lambda\}|dx$$
$$\le \frac{C}{\lambda^p}\int_X \|\varphi_x\|_p^p \le c(d)N^d\frac{C}{\lambda^p}\|f\|_p^p.$$

Let $A = \{(x,s) \in X \times R^d : \sup_{(v,r)\in\Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s+v-t)dt| \ge \lambda\}$. Notice that if $(v,r) \in \Omega_M$, $|s| \le M$ and $|t| < 1/r$, then $f(\tau_{v+s+t}x) = \varphi_x(v+s+t)$ because $3M = N$. Thus,

$$\int_X |\{s : \sup_{(v,r)\in\Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s+v-t)dt| \ge \lambda\}|dx$$
$$\ge \int_R\int_X \chi_A(x,s)\chi_{\{|u|<M\}}(s)dxds$$
$$\ge \int_{|s|<M} m(x : \sup_{(v,r)\in\Omega_M} |H'_r f(\tau_{v+s}x)| \ge \lambda)ds$$
$$= c(d)M^d m(x : \sup_{(v,r)\in\Omega_M} |H'_r f(\tau_v x)| \ge \lambda).$$

Since $N = 3M$, we obtain

$$m(x : \sup_{(v,r)\in\Omega_M} |H'_r f(\tau_v x)| \ge \lambda) \le \frac{3^d C}{\lambda^p}\|f\|_p^p.$$

The proposition follows by letting $M \to \infty$. ■

Corollary 9. *If Ω satisfies the cone condition, then $H'^{\#}_{\Omega} f$ is a weak (1,1) and strong (p,p) operator for $1 < p < \infty$.*

`Proof:` If follows from Corollary 7 and Proposition 8. ■

Theorem 10. *Let $\Omega \subset R^d \times R^+$ satisfy the cone condition and $(0,0) \in \bar{\Omega}$. Then*

$$\lim_{\substack{(v,r)\to(0,0)\\(v,r)\in\Omega}} H'_r f(\tau_v x)$$

exists a.e. for all $f \in L^p(X)$, $1 \le p < \infty$.

`Proof:` It suffices to prove that $k_{v,r}\phi(u) := \int_{r<|t|<1/r} k(t)\phi(u-v-t)dt$ converges in $L^1(R^d)$ as $(v,r) \to (0,0)$, $(v,r) \in \Omega$, for any $\phi \in C_c^1(R^d)$ satisfying $\int_R \phi ds = 0$. Indeed, let

$$O = \{h \in L^1(X) : h(x) = \int g(\tau_t x)\phi(t)dt, g \in L^1(X), \phi \in C_c^1(R^d)\}.$$

Then

$$H'_r h(\tau_v x) = \int g(\tau_s x) k_{v,r} \phi(s) ds.$$

The orthogonal complement of $O \cap L^2(X)$ consists of the invariant functions under the action (see [2]). Thus the theorem would hold for a dense class of functions and then the result follows for all functions by an application of Corollary 9.

Let's introduce some notation

$$K_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if} \quad r \le |s-v|, \\ 0 & \text{otherwise.} \end{cases}$$
$$k_{(v,r)}(s) := \begin{cases} k(s-v) & \text{if} \quad r \le |s-v| \le \frac{1}{r}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} |\int_{r \le |s| \le \frac{1}{r}} f(u-v-s) k(s) ds| &= |\int_{r \le |u-v-s| \le \frac{1}{r}} f(s) k(u-v-s) ds| \\ &= |\int k_{(v,r)}(u-s) f(s) ds| \ = \ |k_{(v,r)} * f(u)| \ . \end{aligned}$$

The L^1-convergence of $k_{(v,r)} * \phi$ follows from the following two properties:
(A) $K_{(v,r)} * \phi$ converges in L^1, and
(B) $\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0$ as $t \to 0$.

Property (A) follows from Lebesgue's Dominated Convergence Theorem. We have that $K_{(v,r)} * \phi$ converges a.e. by Theorem 8. Assume that $\mathrm{supp}(\phi) \subseteq \{|y| \le L\}$, then,

$$|K_{(v,r)} * \phi(u)| \le \Big(c\chi_{\{|y|<2L\}}(u) + \frac{c(d,L)}{|u|^{d+1}} \chi_{R^d \setminus \{|y|<2L\}}(u) \Big) \in L^1(R^d) \ .$$

First consider $|u| \ge 2L$, then using the basic properties of ϕ and $K_{(v,r)}$ (recall (k2)), we can compute (for (v,r) small enough),

$$\begin{aligned} |K_{(v,r)} * \phi(u)| &= \Big| \int [K_{(v,r)}(u-s) - K_{(v,r)}(u)] \phi(s) ds \Big| \\ &\le \int_{|s| \le K} |K_{(v,r)}(u-s) - K_{(v,r)}(u)| \ |\phi(s)| ds \\ &\le \int_{|s| \le K} |k(u-v-s) - k(u-v)| \ |\phi(s)| ds \\ &\le c \int_{|s| \le K} \frac{|s|}{|u-v|^{d+1}} \ |\phi(s)| ds \\ &\le cc(d) \frac{L^{d+1}}{|u|^{d+1}}, \end{aligned}$$

by (k3). Here $c = c(\phi)$.

Consider now $|u| \leq 2L$, hence taking (v,r) small enough

$$|K_{(v,r)} * \phi(u)| = |\int K_{(v,r)}(s)\phi(u-s)ds| \;=\; \Big|\int K_{(v,r)}(s-v)\phi(u+v-s)ds\Big|$$
$$= \Big|\int_{4L\geq|s|\geq r} \frac{w(s)}{|s|^d}\phi(u-v-s)ds\Big|$$
$$c \leq \int_{4L\geq|s|\geq r} \frac{1}{|s|^d}|\phi(u-v-s)-\phi(u-v)|ds \;\leq\; c,$$

where $c = c(\phi)$, because the differential of ϕ is continuous of compact support. This ends the proof of (A).

To prove (B), assume $\mathrm{supp}(\phi) \subseteq \{|y| \leq K\}$. By definition of $K_{(v,r)}$ and $k_{(v,r)}$ we have

$$k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u) \;=\; k_{\frac{1}{r}} * \phi(u-v).$$

Now $K_{\frac{1}{r}} * \phi(u-v) = 0$ if $u \notin S_{(v,r)} := R^d \setminus \{u : |u| < \frac{1}{r} - v - L\}$. We can choose (u,v) small enough such that $u \in S_{(v,r)}$ implies $|u| \geq 2L$, then a similar computation as in (A) gives $|k_{\frac{1}{r}} * \phi(u)| \leq c\frac{1}{|u|^{d+1}}$. In summary

$$|k_{(v,r)} * \phi(u) - K_{(v,r)} * \phi(u)| \leq \chi_{S_{(v,r)}}(u)|k_{\frac{1}{r}} * \phi(u-v)| \leq \chi_{S_{(v,r)}}(u)\frac{c}{|u|^{d+1}}\ .$$

Hence

$$\|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0\ .$$

■

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