A CLASS OF INTEGRAL OPERATORS ON THE UNIT BALL OF $\mathbb{C}^n$

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ABSTRACT. For real parameters a, b, c, and t, where c is not a nonpositive integer, we determine exactly when the integral operator

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{(1-\langle z,w\rangle)^c} f(w)\,dv(w)$$

is bounded on $L^p(\mathbb{B}_n, dv_t)$, where $\mathbb{B}_n$ is the open unit ball in $\mathbb{C}^n$, $1 \le p < \infty$, and $dv_t(z) = (1-|z|^2)^t\,dv(z)$ with dv being volume measure on $\mathbb{B}_n$. The characterization remains the same if we replace $(1-\langle z,w\rangle)^c$ in the integral kernel above by $|1-\langle z,w\rangle|^c$.

1. INTRODUCTION

Throughout the paper we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional complex Euclidean space. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots z_n\overline{w}_n$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} = \sqrt{\langle z, z\rangle}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The boundary of $\mathbb{B}_n$ is the set

$$\mathbb{S}_n = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\},$$

which will be called the unit sphere in $\mathbb{C}^n$.

We denote by dv the volume measure on $\mathbb{B}_n$, normalized so that $v(\mathbb{B}_n) = 1$. For any real parameter t we define

$$dv_t(z) = (1-|z|^2)^t\,dv(z).$$

It is well known that dv_t is finite if and only if $t > -1$.

We are going to study the integral operators $T = T_{a,b,c}$ and $S = S_{a,b,c}$ defined by

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{(1-\langle z,w\rangle)^c} f(w)\,dv(w),$$

and

$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^c} f(w)\,dv(w).$$

Here a, b, and c are real parameters.

We can now state our main results.

Theorem 1. *Suppose $1 < p < \infty$ and c is neither 0 nor a negative integer. Then the following conditions are equivalent.*

(a) *The operator $T_{a,b,c}$ is bounded on $L^p(\mathbb{B}_n, dv_t)$.*
(b) *The operator $S_{a,b,c}$ is bounded on $L^p(\mathbb{B}_n, dv_t)$.*
(c) *The parameters satisfy*

$$\begin{cases} -pa < t+1 < p(b+1) \\ c \le n+1+a+b. \end{cases}$$

Theorem 2. *Suppose $p = 1$ and c is neither 0 nor a negative integer. Then the following conditions are equivalent.*

(a) *The operator $T_{a,b,c}$ is bounded on $L^1(\mathbb{B}_n, dv_t)$.*
(b) *The operator $S_{a,b,c}$ is bounded on $L^1(\mathbb{B}_n, dv_t)$.*
(c) *The parameters satisfy*

$$\begin{cases} -a < t+1 < b+1 \\ c = n+1+a+b. \end{cases} \qquad \text{or} \qquad \begin{cases} -a < t+1 \le b+1 \\ c < n+1+a+b. \end{cases}$$

The special case $n = 1$ and $c = 2 + a + b$ was considered in [3].

The special case $a = 0$ is especially interesting. We denote the resulting operator by $P_{b,c}$. Thus

$$P_{b,c}f(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b f(w)\,dv(w)}{(1-\langle z,w\rangle)^c}.$$

In particular, $P_{b,c}f$ is holomorphic whenever it is defined. This operator is in some sense similar to the Bergman projection, and Theorems 1 and 2 tell us exactly when $P_{b,c}$ maps $L^p(\mathbb{B}_n, dv_t)$ into $L^p(\mathbb{B}_n, dv_s)$, where $1 \le p < \infty$, t is real, and $s > -1$.

2. An Application of Schur's Test

In this section we consider the boundedness of $S_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$ in the case $c = n+1+a+b$. Our estimate is based on the following well-known Schur's test.

Lemma 3. *Let μ be a positive measure on a measure space X, let $H(x,y)$ be a positive measurable function on $X \times X$, and let $p > 1$ with*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If there exists a positive measurable function $h(x)$ on X and if there exists a positive constant C such that

$$\int_X H(x,y)h(y)^q \, d\mu(y) \le Ch(x)^q$$

and

$$\int_X H(x,y)h(x)^p \, d\mu(x) \le Ch(y)^p$$

for all x and y in X, then the integral operator

$$Hf(x) = \int_X H(x,y)f(y)\, d\mu(y)$$

is bounded on $L^p(X,\mu)$ with $\|H\| \le C$.

Proof. See Theorem 3.2.2 of [2]. □

The following estimate will also be crucial to the proof of our main results.

Lemma 4. *Suppose $\alpha > -1$ and s is real. Then the integral*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha \, dv(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+s}}$$

has the following asymptotic behavior as $|z| \to 1^-$.

(a) *If $s < 0$, then $I(z) \sim 1$.*
(b) *If $s = 0$, then $I(z) \sim -\log(1-|z|^2)$.*
(c) *If $s > 0$, then $I(z) \sim (1-|z|^2)^{-s}$.*

Proof. See Proposition 1.4.10 of [1]. □

The following simple fact will be used many times later in the paper, so we collect it here for convenience of reference.

Lemma 5. *The measure $dv_t(z) = (1-|z|^2)^t \, dv(z)$ is finite on $\mathbb{B}_n$ if and only if $t > -1$.*

Proof. This follows easily from polar coordinates; see 1.4.3 of [1]. □

We now prove the main esitmate of this section.

Lemma 6. *Suppose* $1 \le p < \infty$, $-pa < t+1 < p(b+1)$, *and* $c = n+1+a+b$. *Then the operator* $S = S_{a,b,c}$ *is bounded on* $L^p(\mathbb{B}_n, dv_t)$.

Proof. If $p=1$ and $-a < t+1 < b+1$, then for every $f \in L^1(\mathbb{B}_n, dv_t)$ we can apply Fubini's theorem to obtain

$$\begin{aligned} &\int_{\mathbb{B}_n} |Sf(z)|\, dv_t(z) \\ \le\ & \int_{\mathbb{B}_n} (1-|z|^2)^{a+t}\, dv(z) \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b |f(w)|\, dv(w)}{|1-\langle z,w\rangle|^{n+1+a+b}} \\ =\ & \int_{\mathbb{B}_n} (1-|w|^2)^b |f(w)|\, dv(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\, dv(z)}{|1-\langle z,w\rangle|^{n+1+a+b}} \\ =\ & \int_{\mathbb{B}_n} (1-|w|^2)^b |f(w)|\, dv(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\, dv(z)}{|1-\langle z,w\rangle|^{n+1+a+t+(b-t)}}. \end{aligned}$$

Since $a+t > -1$ and $b-t > 0$, we deduce from part (c) of Lemma 4 that there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\, dv(z)}{|1-\langle z,w\rangle|^{n+1+a+t+(b-t)}} \le \frac{C}{(1-|w|^2)^{b-t}}$$

for all $w \in \mathbb{B}_n$. It follows that

$$\int_{\mathbb{B}_n} |Sf(z)|\, dv_t(z) \le C \int_{\mathbb{B}_n} |f(w)|\, dv_t(w),$$

and so S is bounded on $L^1(\mathbb{B}_n, dv_t)$.

If $1 < p < \infty$ and $1/p + 1/q = 1$, then the inequalities

$$-pa < t+1 < p(b+1)$$

is equivalent to

$$-\frac{b+1}{q} < \frac{b-t}{p}, \qquad -\frac{a+t+1}{p} < \frac{a}{q}.$$

These two inequalities clearly imply that

$$\left(-\frac{b+1}{q}, \frac{a}{q}\right) \bigcap \left(-\frac{a+t+1}{p}, \frac{b-t}{p}\right)$$

is nonempty. Pick any number s from the above intersection and let

$$h(z) = (1-|z|^2)^s, \qquad z \in \mathbb{B}_n.$$

We can write the integral operator $S = S_{a,b,c}$ as

$$Sf(z) = \int_{\mathbb{B}_n} H(z,w) f(w)\, dv_t(w),$$

where

$$H(z,w)=\frac{(1-|z|^2)^a(1-|w|^2)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}}.$$

It follows from Lemmas 3 and 4 that the integral operator S is bounded on $L^p(\mathbb{B}_n, dv_t)$. ☐

3. SUFFICIENCY FOR THE BOUNDEDNESS OF $S_{a,b,c}$

In this section we obtain sufficient conditions for the boundedness of the operator $S = S_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$. In the next section we shall show that these conditions are also necessary.

Lemma 7. *If $1 \le p < \infty$, $-pa < t+1 < p(b+1)$, and $c \le n+1+a+b$, then the operator $S_{a,b,c}$ is bounded on $L^p(\mathbb{B}_n, dv_t)$.*

Proof. Write

$$\sigma = (n+1+a+b) - c.$$

Then $\sigma \ge 0$ and

$$\frac{1}{|1-\langle z,w\rangle|^c} = \frac{|1-\langle z,w\rangle|^\sigma}{|1-\langle z,w\rangle|^{n+1+a+b}} \le \frac{2^\sigma}{|1-\langle z,w\rangle|^{n+1+a+b}}.$$

Combining this with Lemma 6, we conclude that the operator S is bounded on $L^p(\mathbb{B}_n, dv_t)$. ☐

Lemma 8. *If $p = 1$, $-a < t+1 = b+1$, and $c < n+1+a+b$, then the operator $S = S_{a,b,c}$ is bounded on $L^1(\mathbb{B}_n, dv_t)$.*

Proof. We apply Fubini's theorem again to obtain

$$\begin{aligned}
&\int_{\mathbb{B}_n} |Sf(z)|\,dv_t(z)\\
\le\ &\int_{\mathbb{B}_n} (1-|w|^2)^b |f(w)|\,dv(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\,dv(z)}{|1-\langle z,w\rangle|^c}\\
=\ &\int_{\mathbb{B}_n} |f(w)|\,dv_t(w) \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\,dv(z)}{|1-\langle z,w\rangle|^c}.
\end{aligned}$$

By assumption, $a+t > -1$ and $c = n+1+(a+t)+s$, where

$$s = c-(n+1+a+t) = c-(n+1+a+b) < 0.$$

By Lemma 4, there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{a+t}\,dv(z)}{|1-\langle z,w\rangle|^c} \le C$$

for all $w \in \mathbb{B}_n$. This shows that the operator S is bounded on $L^1(\mathbb{B}_n, dv_t)$. ☐

4. Necessity for the Boundedness of $T_{a,b,c}$

In this section we obtain necessary conditions for the boundedness of the operator $T = T_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$. These conditions turn out to be the same as the sufficient conditions we obtained in the previous section for the boundedness of the operator $S = S_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$.

Lemma 9. *Suppose* $1 \le p < \infty$ *and* $T = T_{a,b,c}$ *is bounded on* $L^p(\mathbb{B}_n, dv_t)$. *Then* $-pa < t + 1$.

Proof. Recall that

$$Tf(z) = (1 - |z|^2)^a \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^b}{(1 - \langle z, w\rangle)^c} f(w)\, dv(w).$$

Choose a positive number N such that $Np + t > -1$ and $N + b > -1$. It follows from Lemma 5 that the function

$$f_N(z) = (1 - |z|^2)^N$$

belongs to $L^p(\mathbb{B}_n, dv_t)$. Since the kernel $(1 - \langle z, w\rangle)^c$ is anti-holomorphi in w, an application of the symmetry of $\mathbb{B}_n$ shows that there exists a constant $C_N > 0$ such that

$$Tf_N(z) = C_N(1 - |z|^2)^a, \qquad z \in \mathbb{B}_n.$$

Since Tf_N belongs to $L^p(\mathbb{B}_n, dv_t)$, it follows from Lemma 5 that $pa + t > -1$, or $-pa < t + 1$. □

Lemma 10. *Suppose* $1 \le p < \infty$ *and* $T = T_{a,b,c}$ *is bounded on* $L^p(\mathbb{B}_n, dv_t)$. *Then* $t + 1 \le p(b + 1)$, *and strict inequality holds when* $1 < p < \infty$.

Proof. Let q be the conjugate of p. Thus $1/p + 1/q = 1$ when $1 < p < \infty$ and $q = \infty$ when $p = 1$. By duality, the boundedness of T on $L^p(\mathbb{B}_n, dv_t)$ implies the boundedness of T^* on $L^q(\mathbb{B}_n, dv_t)$. It is easy to see that, with respect to the duality $L^p(\mathbb{B}_n, dv_t)^* = L^q(\mathbb{B}_n, dv_t)$, we have

$$T^*f(z) = (1 - |z|^2)^{b-t} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{a+t}}{(1 - \langle z, w\rangle)^c} f(w)\, dv(w).$$

If $1 < p < \infty$, then the boundedness of T^* on $L^q(\mathbb{B}_n, dv_t)$ implies

$$q(b - t) + t > -1;$$

see Lemma 9 or its proof. It is easy to see that the inequality above is equivalent to

$$t + 1 < p(b + 1).$$

If $p = 1$, so the operator T^* is bounded on $L^\infty(\mathbb{B}_n)$, then we can apply T^* to a bounded function of the form

$$f_N(z) = (1 - |z|^2)^N,$$

where N is a sufficiently large positive number, to obtain a bounded function T^*f_N. Once again, it is easy to see that there exists a positive constant C_N such that

$$T^*f_N(z) = C_N(1-|z|^2)^{b-t}, \qquad z \in \mathbb{B}_n.$$

The boundedness of this function clearly implies that $b - t \geq 0$, or

$$t + 1 \leq 1(b+1).$$

This completes the proof of the lemma. □

The next lemma shows that, in the special case $c = n + 1 + a + b$, strict inequality in $t + 1 \leq p(b + 1)$ must also hold even when $p = 1$.

Lemma 11. *Suppose* $c = n + 1 + a + b$ *and* $T = T_{a,b,c}$ *is bounded on* $L^1(\mathbb{B}_n, dv_t)$. *Then* $t < b$.

Proof. By the duality argument used in the proof of the previous lemma, the operator

$$T^*f(z) = (1-|z|^2)^{b-t}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{a+t}}{(1-\langle z,w\rangle)^{n+1+a+b}}f(w)\,dv(w)$$

is bounded on $L^\infty(\mathbb{B}_n)$. For any point $a \in \mathbb{B}_n$ consider the function

$$f_a(z) = \frac{(1-\langle a,z\rangle)^{n+1+a+b}}{|1-\langle a,z\rangle|^{n+1+a+b}}, \qquad z \in \mathbb{B}_n.$$

It is obvious that $\|f_a\|_\infty = 1$ for every $a \in \mathbb{B}_n$.

On the other hand, we have

$$\|T^*f_a\|_\infty \geq |T^*f_a(a)| = (1-|a|^2)^{b-t}\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{a+t}dv(w)}{|1-\langle a,w\rangle|^{n+1+a+b}}.$$

If $b = t$, then an application of part (b) of Lemma 4 shows that

$$\lim_{|a|\to 1^-}\|T^*f_a\|_\infty = \infty,$$

a contradiction to the assumption that T^* is bounded on $L^\infty(\mathbb{B}_n)$. Since Lemma 10 tells us that $t > b$ is impossible, we must have $t < b$. □

Lemma 12. *Suppose* $1 \leq p < \infty$ *and the operator* $T = T_{a,b,c}$ *is bounded on* $L^p(\mathbb{B}_n, dv_t)$. *If* c *is not a nonpositive integer, then* $c \leq n + 1 + a + b$.

Proof. We consider the functions $f_{N,k}$ defined by

$$f_{N,k}(z) = (1-|z|^2)^N z_1^k, \qquad z \in \mathbb{B}_n,$$

where k is a positive integer and N is large enough so that $N + b > -1$ and $Np + t > -1$.

We first estimate the norm of $f_{N,k}$ in $L^p(\mathbb{B}_n, dv_t)$:

$$\|f_{N,k}\|_t^p = \int_{\mathbb{B}_n} |f_{N,k}(z)|^p \, dv_t(z) = \int_{\mathbb{B}_n} (1-|z|^2)^{Np+t}|z_1|^{pk} \, dv(z).$$

For $z \in \mathbb{B}_n$ we write $z = r\zeta$, where $0 \le r < 1$ and $\zeta \in \mathbb{S}_n$, and integrate in polar coordinates (see 1.4.3 of [1]) to obtain

$$\|f_{N,k}\|_t^p = 2n \int_0^1 (1-r^2)^{Np+t} r^{2n+pk-1} \, dr \int_{\mathbb{S}_n} |\zeta_1|^{pk} \, d\sigma(\zeta),$$

where $d\sigma$ is the normalized surface measure on $\mathbb{S}_n$. The radial integral above is equal to

$$\begin{aligned} n\int_0^1 (1-r)^{Np+t} r^{n+\frac{pk}{2}-1} \, dr &= nB\left(Np+t+1, n+\frac{pk}{2}\right) \\ &= n\frac{\Gamma(Np+t+1)\Gamma\left(n+\frac{pk}{2}\right)}{\Gamma\left(Np+\frac{pk}{2}+n+t+1\right)}. \end{aligned}$$

We are going to fix N but let $k \to \infty$. By Stirling's formula, the radial integral above is comparable to $k^{-(Np+t+1)}$ as $k \to \infty$. In particular, there exists a constant $C_1 > 0$, independent of k, such that

$$\|f_{N,k}\|_t^p \le \frac{C_1}{k^{Np+t+1}} \int_{\mathbb{S}_n} |\zeta_1|^{pk} \, d\sigma(\zeta) \tag{1}$$

for all $k \ge 1$.

We next compute the norm of $Tf_{N,k}$.

Since c is neither 0 nor a negative integer, we use the Taylor expansion for $(1-\langle z,w\rangle)^{-c}$ and the multi-nomial expansion for $\langle z,w\rangle^k$ to obtain

$$\begin{aligned} Tf_{N,k}(z) &= (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{N+b} w_1^k \, dv(w)}{(1-\langle z,w\rangle)^c} \\ &= (1-|z|^2)^a \int_{\mathbb{B}_n} (1-|w|^2)^{N+b} w_1^k \sum_{m=0}^{\infty} \frac{\Gamma(m+c)}{m!\,\Gamma(c)} \langle z,w\rangle^m \, dv(w) \\ &= \frac{\Gamma(k+c)}{k!\,\Gamma(c)} (1-|z|^2)^a \int_{\mathbb{B}_n} (1-|w|^2)^{N+b} w_1^k \langle z,w\rangle^k \, dv(w) \\ &= \frac{\Gamma(k+c)}{k!\,\Gamma(c)} (1-|z|^2)^a z_1^k \int_{\mathbb{B}_n} (1-|w|^2)^{N+b} |w_1|^{2k} \, dv(w). \end{aligned}$$

By polar coordinates and Proposition 1.4.9 of [1], the last integral above is equal to

$$\begin{aligned} & 2n\int_0^1 (1-r^2)^{N+b} r^{2n+2k-1}\,dr \int_{\mathbb{S}_n} |\zeta_1|^{2k}\,d\sigma(\zeta) \\ = \;& n\int_0^1 (1-r)^{N+b} r^{n+k-1}\,dr \int_{\mathbb{S}_n} |\zeta_1^k|^2\,d\sigma(\zeta) \\ = \;& n\frac{\Gamma(N+b+1)\Gamma(n+k)}{\Gamma(N+b+1+n+k)}\frac{(n-1)!\,k!}{(n-1+k)!}. \end{aligned}$$

It follows that

$$Tf_{N,k}(z) = \frac{n!\,\Gamma(N+b+1)\Gamma(k+c)}{\Gamma(c)\Gamma(N+b+1+n+k)}(1-|z|^2)^a z_1^k.$$

By Stirling's formula,

$$\frac{n!\,\Gamma(N+b+1)\Gamma(k+c)}{\Gamma(c)\Gamma(N+b+1+n+k)} \sim \frac{1}{k^{N+b+1+n-c}}$$

as $k\to\infty$, and by the arguments in the first paragraph of this proof (with a in place of N),

$$\int_{\mathbb{B}_n} (1-|z|^2)^{pa}|z_1|^{pk}\,dv_t(z) \sim \frac{1}{k^{pa+t+1}}\int_{\mathbb{S}_n} |\zeta_1|^{pk}\,d\sigma(\zeta)$$

as $k\to\infty$. So we can find a constant $C_2>0$, independent of k, such that

$$\|Tf_{N,k}\|_t^p \geq \frac{C_2}{k^{p(N+b+1+n-c)+(pa+t+1)}} \tag{2}$$

for all $k\geq 1$. Combining (1) and (2) with the boundedness of T on $L^p(\mathbb{B}_n, dv_t)$, we obtain a positive constant C, independent of k, such that

$$\frac{1}{k^{pN+pb+p+pn-pc+pa+t+1}} \leq \frac{C}{k^{Np+t+1}},$$

or

$$\frac{1}{k^{p(n+1+a+b-c)}} \leq C, \qquad k\geq 1.$$

This is possible only when $c\leq n+1+a+b$. □

5. COMLETING THE PROOF OF THEOREMS 1 AND 2

We now put all the pieces together to prove the two main theorems stated in the introduction.

It is obvious that the boundedness of $S_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$ implies the boundedness of $T_{a,b,c}$ on $L^p(\mathbb{B}_n, dv_t)$. So (b) implies (a) in both Theorem 1 and Theorem 2.

That (a) implies (c) in Theorem 1 follows from Lemma 9, Lemma 10, and Lemma 12. That (a) implies (c) in Theorem 2 follows from Lemmas 9-12.

It follows from Lemma 7 that (c) implies (b) in Theorem 1. That (c) implies (b) in Theorem 2 follows from Lemma 7 and Lemma 8.

This completes the proof of Theorems 1 and 2.

6. An Application

In this section we apply the main result proved earlier to characterize a class of Banach spaces of holomorphic functions in $\mathbb{B}_n$, including the weighted Bergman spaces and the holomorphic Besov spaces of $\mathbb{B}_n$.

Throughout this section we use

$$m = (m_1, m_2, \cdots, m_n)$$

to denote a multi-index of nonnegative integers. It is common practice to write

$$|m| = m_1 + m_2 + \cdots + m_n, \qquad m! = m_1!m_2! \cdots m_n!.$$

If $z \in \mathbb{B}_n$, we write

$$z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}.$$

Similarly, if f is holomorphic in $\mathbb{B}_n$, we write

$$\partial^m f(z) = \frac{\partial^{|m|} f}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}}.$$

In this section we are going to modify the definition of dv_t as follows:

$$dv_t(z) = c_t(1 - |z|^2)^t \, dv(z),$$

where $c_t = 1$ for $t \le -1$, and for $t > -1$, c_t is chosen so that dv_t is a probability measure. This slight abuse of notation is clearly harmless, but it will simplify our presentation in many instances.

For $\alpha > -1$ and $p > 0$ we use

$$A^p_\alpha = H(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, dv_\alpha)$$

to denote the weighted Bergman space, where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. It is well known that A^p_α is a closed subspace of $L^p(\mathbb{B}_n, dv_\alpha)$. In particular, there exists an orthogonal projection

$$P_\alpha : L^2(\mathbb{B}_n, dv_\alpha) \to A^2_\alpha.$$

This will be called the (weighted) Bergman projection. It is well known that P_α is an integral operator. More specifically,

$$P_\alpha f(z) = \int_{\mathbb{B}_n} \frac{f(w) \, dv_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

This integral representation can be used to extend the domain of P_α to $L^1(\mathbb{B}_n, dv_\alpha)$.

The main result of this section is the following.

Theorem 13. *Suppose β is real, $\alpha > -1$, $p \geq 1$, and N is a positive integer satisfying*

$$-pN < \beta + 1 < p(\alpha + 1).$$

If f is holomorphic in $\mathbb{B}_n$, then $f \in P_\alpha L^p(\mathbb{B}_n, dv_\beta)$ if and only if the functions $(1 - |z|^2)^N \partial^m f(z)$, where $|m| = N$, all belong to $L^p(\mathbb{B}_n, dv_\beta)$.

Proof. If $f = P_\alpha(g)$ for some $g \in L^p(\mathbb{B}_n, dv_\beta)$, then

$$f(z) = c_\alpha \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha g(w)\, dv(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}.$$

For each multi-index m with $|m| = N$ we differentiate under the integral sign to obtain

$$(1 - |z|^2)^N \partial^m f(z) = C_m (1 - |z|^2)^N \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha \overline{w}^m g(w)\, dv(w)}{(1 - \langle z, w\rangle)^{n+1+N+\alpha}},$$

where C_m is a positive constant depending on m. An application of Theorems 1 and 2 shows that the functions

$$(1 - |z|^2)^N \partial^m f(z), \qquad |m| = N,$$

all belong to $L^p(\mathbb{B}_n, dv_\beta)$.

On the other hand, if the functions $(1 - |z|^2)^N \partial^m f(z)$ are in $L^p(\mathbb{B}_n, dv_\beta)$ for every multi-index m with $|m| = N$, then the functions $(1-|z|^2)^N \partial^m f(z)$ belong to $L^p(\mathbb{B}_n, dv_\beta)$ for every multi-index m with $|m| \leq N$; see Theorem 2.17 of [4]. Consider the function

$$g(z) = C(1 - |z|^2)^N R^{\alpha,N} f(z),$$

where C is an appropriate constant and $R^{\alpha,N}$ is the radial differential operator defined in [4]. By Proposition 1.15 of [4], $R^{\alpha,N}$ is a linear partial differential operator on $H(\mathbb{B}_n)$ of order N with polynomial coefficients. Therefore, the assumption on f ensures that the function g belongs to $L^p(\mathbb{B}_n, dv_\beta)$. Now

$$P_\alpha(g)(z) = CC_\alpha \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{N+\alpha} R^{\alpha,N} f(w)\, dv(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}.$$

By Proposition 1.14 of [4], we can choose the constant C such that

$$P_\alpha(g)(z) = R_{\alpha,N} R^{\alpha,N} f(z) = f(z),$$

where $R_{\alpha,N}$ is the radial integral operator defined in [4], which is just the inverse of the operator $R^{\alpha,N}$. This completes the proof of the theorem. □

An interesting special case is when $\beta = -(n + 1)$. In this case, the theorem above characterizes the holomorphic Besov space B_p, $1 \leq p < \infty$, as the image of the Lebesgue space $L^p(\mathbb{B}_n, d\tau)$ under the weighted Bergman projection P_α, where

$$d\tau(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the Möbius invariant measure on $\mathbb{B}_n$. See [4] for more information on the Besov spaces B_p.

REFERENCES

[1] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.
[2] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.
[3] K. Zhu, A Forelli-Rudin type theorem with applications, *Complex Variables* **16** (1991), 107-113.
[4] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2004.

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