

LIPSCHITZ TYPE CHARACTERIZATIONS FOR BERGMAN SPACES

HASI WULAN AND KEHE ZHU

ABSTRACT. We obtain new characterizations for Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. As a consequence, we prove optimal embedding theorems when an analytic function on the unit disk is symmetrically lifted to the bidisk.

1. INTRODUCTION

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} . For any $\alpha > -1$ we consider the weighted area measure

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

where dA is the normalized area measure on \mathbb{D} . It is easy to see that each dA_α is a probability measure on \mathbb{D} .

For $p > 0$ and $\alpha > -1$ we denote by A_α^p the space of analytic functions f in \mathbb{D} such that

$$\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

These are called weighted Bergman spaces with standard weights. See [1] and [6] for the modern theory of Bergman spaces.

Three different metrics on the unit disk will be used in the paper. First, the usual Euclidean metric is of course written as $|z - w|$. Second, the pseudo-hyperbolic metric on \mathbb{D} is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|.$$

And finally, the hyperbolic metric on \mathbb{D} is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

We mention that the hyperbolic metric is also called the Bergman metric, and sometimes the Poincare metric as well.

2000 *Mathematics Subject Classification.* 32A36.

Key words and phrases. Bergman spaces, hyperbolic metric, Lipschitz condition.

Research partially supported by the National Science Foundations of the US and China.

The main result of the paper is the following.

Theorem 1. *Suppose $p > 0$, $\alpha > -1$, and f is analytic in \mathbb{D} . Then the following conditions are equivalent.*

- (a) f belongs to A_α^p .
- (b) There exists a continuous function g in $L^p(\mathbb{D}, dA_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

- (c) There exists a continuous function g in $L^p(\mathbb{D}, dA_\alpha)$ such that

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

- (d) There exists a continuous function g in $L^p(\mathbb{D}, dA_{p+\alpha})$ such that

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w))$$

for all z and w in \mathbb{D} .

Note that the same measure dA_α appears in conditions (a), (b), and (c), but condition (d) involves a different measure $dA_{p+\alpha}$.

Similar characterizations for Hardy-Sobolev spaces have appeared in the literature before. See [4][5] for example. The present paper was motivated by [9]. As another motivation for our result, we mention that the classical Bloch space \mathcal{B} , consisting of analytic functions f in \mathbb{D} such that

$$\sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty,$$

also admits a Lipschitz type characterization. More specifically, an analytic function f in \mathbb{D} belongs to \mathcal{B} if and only if there exists a positive constant C such that

$$|f(z) - f(w)| \leq C\beta(z, w)$$

for all z and w in \mathbb{D} ; see [6] or [12] for example. It is then clear that an analytic function f in \mathbb{D} belongs to \mathcal{B} if and only if there exists a continuous function g in $L^\infty(\mathbb{D})$ such that

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

Since the Bergman metric is based on the reproducing kernel of the Bergman space, it is no surprise that our Lipschitz type characterizations for weighted Bergman spaces appear more natural when the Bergman metric (and its bounded counterpart, the pseudo-hyperbolic metric) is used. Although a characterization in terms of the Euclidean metric (condition (d)) is possible, it gives one the impression of something artificial.

2. PRELIMINARIES

We collect some preliminary results in this section that involve the hyperbolic and pseudo-hyperbolic metrics.

For any $0 < r < 1$ and $z \in \mathbb{D}$ we let

$$D(z, r) = \{w \in \mathbb{D} : \rho(w, z) < r\}$$

denote the pseudo-hyperbolic disk centered at z with radius r . It is well known that $D(z, r)$ is actually a Euclidean disk with Euclidean center and Euclidean radius given by

$$\frac{1 - r^2}{1 - r^2|z|^2} z, \quad \frac{1 - |z|^2}{1 - r^2|z|^2} r,$$

respectively; see [3] for example. In particular, if r is fixed, then the area of $D(z, r)$ is comparable to $(1 - |z|^2)^2$.

The pseudo-hyperbolic metric is bounded by 1. But the hyperbolic metric is unbounded. For any $R > 0$ and $z \in \mathbb{D}$ we let

$$E(z, R) = \{w \in \mathbb{D} : \beta(w, z) < R\}$$

denote the hyperbolic disk centered at z with radius R . If $0 < r < 1$ and

$$(1) \quad R = \frac{1}{2} \log \frac{1+r}{1-r},$$

then it is clear that $E(z, R) = D(z, r)$. Consequently, if R is fixed, then the area of $E(z, R)$ is comparable to $(1 - |z|^2)^2$ as well. By the same token, any estimate in terms of the pseudo-hyperbolic metric can be translated to one in terms of the hyperbolic metric, and vice versa.

Lemma 2. *For any fixed $r \in (0, 1)$ there exists a positive constant C such that*

$$C^{-1} \leq \frac{1 - |z|^2}{|1 - \bar{z}w|} \leq C$$

whenever $\rho(z, w) \leq r$. Consequently, there exists a positive constant C such that

$$C^{-1} \leq \frac{1 - |z|^2}{1 - |w|^2} \leq C$$

whenever $\rho(z, w) \leq r$.

Proof. This is well known. See [6] or [12] for example. □

Lemma 3. *For any fixed $r \in (0, 1)$ and there exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^2} \int_{D(z, r)} |f(w)|^p dA(w)$$

for all $z \in \mathbb{D}$, all $p > 0$, and all analytic f in \mathbb{D} .

Proof. This is well known as well. See [6] or [12] for example. \square

Lemma 4. For any fixed $z \in \mathbb{D}$ we have

$$\lim_{w \rightarrow z} \frac{\beta(z, w)}{|z - w|} = \lim_{w \rightarrow z} \frac{\rho(z, w)}{|z - w|} = \frac{1}{1 - |z|^2}.$$

Proof. This follows from elementary calculations. \square

Lemma 5. For any $\alpha > -1$ and $p > 0$ there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \leq C \left[|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) \right]$$

and

$$|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) \leq C \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all analytic functions f in \mathbb{D} , where

$$g(z) = (1 - |z|^2)f'(z)$$

for $z \in \mathbb{D}$.

Proof. See [6] or [12] for example. \square

3. THE MAIN RESULT

We now prove the main result of the paper. The proof is constructive in the sense that we will actually produce a formula for the function g that appears in various conditions of Theorem 1. We prove Theorem 1 as three separate results.

Theorem 6. Suppose $p > 0$, $\alpha > -1$, and f is analytic in \mathbb{D} . Then $f \in A_\alpha^p$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that

$$(2) \quad |f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

Proof. First assume that condition (2) holds. Then

$$\left| \frac{f(z) - f(w)}{z - w} \right| \leq \frac{\rho(z, w)}{|z - w|} (g(z) + g(w))$$

for all $z \neq w$ in \mathbb{D} . Fix any $z \in \mathbb{D}$, let $w \rightarrow z$, and use Lemma 4. We obtain

$$(1 - |z|^2)|f'(z)| \leq 2g(z), \quad z \in \mathbb{D}.$$

Since $g \in L^p(\mathbb{D}, dA_\alpha)$, an application of Lemma 5 shows that $f \in A_\alpha^p$.

Next assume that $f \in A_\alpha^p$. We are going to produce a continuous function g that satisfies condition (2). To this end we fix a radius $r \in (0, 1)$ and

consider any two points z and w in \mathbb{D} with $\rho(z, w) < r$. By the fundamental theorem of calculus,

$$f(z) - f(w) = (z - w) \int_0^1 f'(tz + (1-t)w) dt.$$

Since z and w are points in the convex set $D(z, r)$, we see that

$$tz + (1-t)w \in D(z, r)$$

for all $t \in [0, 1]$. It follows that

$$|f(z) - f(w)| \leq |z - w| \sup\{|f'(u)| : u \in D(z, r)\}.$$

By Lemma 2, there exists a positive constant C that only depends on r such that

$$|f(z) - f(w)| \leq \rho(z, w)h(z),$$

where

$$h(z) = C \sup\{(1 - |u|^2)|f'(u)| : u \in D(z, r)\}, \quad z \in \mathbb{D}.$$

Obviously, we also have

$$|f(z) - f(w)| \leq \rho(z, w)(h(z) + h(w)),$$

provided that $\rho(z, w) < r$.

If $\rho(z, w) \geq r$, then clearly

$$|f(z) - f(w)| \leq \frac{\rho(z, w)}{r} (|f(z)| + |f(w)|).$$

If we define

$$g(z) = \frac{|f(z)|}{r} + h(z), \quad z \in \mathbb{D},$$

then we have

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

It is clear that the function g above is continuous on \mathbb{D} . It remains for us to show that $g \in L^p(\mathbb{D}, dA_\alpha)$. Since f is already in $L^p(\mathbb{D}, dA_\alpha)$, it suffices for us to show that $h \in L^p(\mathbb{D}, dA_\alpha)$.

Recall that if r and R are related as in (1), then $D(z, r) = E(z, R)$. So we can choose $r' \in (0, 1)$ so that $D(z, r') = E(z, 2R)$ for all $z \in \mathbb{D}$. By the triangle inequality for the Bergman metric β we have $E(u, R) \subset E(z, 2R)$ whenever $u \in E(z, R)$. Equivalently, we have $D(u, r) \subset D(z, r')$ whenever $u \in D(z, r)$. It follows from this and Lemma 3 that there exists a constant $C > 0$ that only depends on r such that

$$h(z)^p \leq \frac{C}{(1 - |z|^2)^{2-p}} \int_{D(z, r')} |f'(w)|^p dA(w)$$

for all $z \in \mathbb{D}$. If we use χ_z to denote the characteristic function of the set $D(z, r')$, then clearly $\chi_z(w) = \chi_w(z)$, and

$$h(z)^p \leq \frac{C}{(1 - |z|^2)^{2-p}} \int_{\mathbb{D}} |f'(w)|^p \chi_z(w) dA(w)$$

for all $z \in \mathbb{D}$. Writing $C_1 = (\alpha + 1)C/\pi$ and using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{D}} h(z)^p dA_\alpha(z) &\leq C_1 \int_{\mathbb{D}} (1 - |z|^2)^{p+\alpha-2} dA(z) \int_{\mathbb{D}} |f'(w)|^p \chi_z(w) dA(w) \\ &= C_1 \int_{\mathbb{D}} |f'(w)|^p dA(w) \int_{\mathbb{D}} (1 - |z|^2)^{p+\alpha-2} \chi_w(z) dA(z) \\ &= C_1 \int_{\mathbb{D}} |f'(w)|^p dA(w) \int_{D(w, r')} (1 - |z|^2)^{p+\alpha-2} dA(z). \end{aligned}$$

Combining this with Lemma 2 and the fact that the area of $D(w, r')$ is comparable to $(1 - |w|^2)^2$, we obtain another positive constant C_2 , which only depends on α and r , such that

$$\int_{\mathbb{D}} h(z)^p dA_\alpha(z) \leq C_2 \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w).$$

In view of Lemma 5, this shows that $h \in L^p(\mathbb{D}, dA_\alpha)$ and completes the proof of Theorem 6. \square

Theorem 7. *Suppose $p > 0$, $\alpha > -1$, and f is analytic in \mathbb{D} . Then $f \in A_\alpha^p$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that*

$$(3) \quad |f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{D} .

Proof. If condition (3) is satisfied, we divide both sides of (3) by $|z - w|$ and, with the help of Lemma 4, take the limit as $w \rightarrow z$. The result is

$$(1 - |z|^2)|f'(z)| \leq 2g(z), \quad z \in \mathbb{D}.$$

This along with Lemma 5 shows that $f \in A_\alpha^p$.

If $f \in A_\alpha^p$, then by Theorem 6, there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that condition (2) holds. Since $\rho(z, w) \leq \beta(z, w)$ for all z and w in \mathbb{D} , the same function g also satisfies condition (3). This completes the proof of Theorem 7. \square

Theorem 8. *Suppose $p > 0$, $\alpha > -1$, and f is analytic in \mathbb{D} . Then $f \in A_\alpha^p$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_{p+\alpha})$ such that*

$$(4) \quad |f(z) - f(w)| \leq |z - w|(g(z) + g(w))$$

for all z and w in \mathbb{D} .

Proof. If condition (4) holds, we divide both sides of (4) by $|z - w|$ and take the limit as $w \rightarrow z$. The result is $|f'(z)| \leq 2g(z)$, so that

$$(1 - |z|^2)|f'(z)| \leq 2(1 - |z|^2)g(z)$$

for all $z \in \mathbb{D}$. Since $g \in L^p(\mathbb{D}, dA_{p+\alpha})$, we see that the function

$$(1 - |z|^2)f'(z)$$

belongs to $L^p(\mathbb{D}, dA_\alpha)$, which, according to Lemma 5, means that $f \in A_\alpha^p$.

If $f \in A_\alpha^p$, then by Theorem 6, there exists a continuous function h in $L^p(\mathbb{D}, dA_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(h(z) + h(w))$$

for all z and w in \mathbb{D} . Rewrite this as

$$|f(z) - f(w)| \leq |z - w| \left[\frac{h(z)}{|1 - \bar{z}w|} + \frac{h(w)}{|1 - \bar{z}w|} \right],$$

and apply the triangle inequalities

$$|1 - \bar{z}w| \geq 1 - |z|, \quad |1 - \bar{z}w| \geq 1 - |w|.$$

We obtain

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{D},$$

where

$$g(z) = \frac{h(z)}{1 - |z|} \leq \frac{2h(z)}{1 - |z|^2}.$$

Since $h \in L^p(\mathbb{D}, dA_\alpha)$, we have $g \in L^p(\mathbb{D}, dA_{p+\alpha})$. This completes the proof of Theorem 8. \square

4. LIFTING FUNCTIONS FROM THE DISK TO THE BIDISK

Let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ denote the bidisk in \mathbb{C}^2 and let $H(\mathbb{D}^2)$ denote the space of all holomorphic functions in \mathbb{D}^2 . Similarly, $H(\mathbb{D})$ is the space of all analytic functions in \mathbb{D} . For $p > 0$ and $\alpha > -1$ we define $A_\alpha^p(\mathbb{D}^2)$ as the space of functions $f \in H(\mathbb{D}^2)$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |f(z, w)|^p dA_\alpha(z) dA_\alpha(w) < \infty.$$

These are also called weighted Bergman spaces.

In this section we present an application of our main theorem to the problem of lifting analytic functions from the unit disk to the bidisk. Thus we consider the symmetric lifting operator

$$L : H(\mathbb{D}) \rightarrow H(\mathbb{D}^2)$$

defined by

$$L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.$$

We will also need the associated diagonal operator

$$\Delta : H(\mathbb{D}^2) \rightarrow H(\mathbb{D})$$

which is defined by

$$\Delta(f)(z) = f(z, z).$$

The action of the diagonal operator on Hardy and Bergman spaces of the polydisk has been studied by several authors. See [2] [7][10][11]. In particular, the diagonal operator has the following property.

Lemma 9. *Suppose $p > 0$ and $\alpha > -1$. Then the operator Δ maps $A_\alpha^p(\mathbb{D}^2)$ boundedly onto $A_{2(\alpha+1)}^p(\mathbb{D})$.*

Proof. See [10] or [11]. □

The following standard estimate will be needed in the proof of our lifting theorems.

Lemma 10. *Suppose $s > -1$, t is real, and*

$$I(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^s dA(z)}{|1 - \bar{z}w|^{2+s+t}}, \quad z \in \mathbb{D}.$$

Then $I(z)$ is bounded in \mathbb{D} whenever $t < 0$; and $I(z)$ is bounded by $(1 - |z|^2)^{-t}$ whenever $t > 0$.

Proof. See [6] or [12]. □

We now obtain the first lifting theorem.

Theorem 11. *Suppose $\alpha > -1$ and $0 < p < \alpha + 2$. Then the symmetric lifting operator L maps $A_\alpha^p(\mathbb{D})$ boundedly into $A_\alpha^p(\mathbb{D}^2)$. Moreover, this is no longer true when $p > \alpha + 2$.*

Proof. Given $f \in A_\alpha^p(\mathbb{D})$, we apply Theorem 6 to find a function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that condition (2) holds. Then there exists a constant $C = C_p$ such that

$$(5) \quad |L(f)(z, w)|^p \leq C \left[\frac{g(z)^p}{|1 - \bar{z}w|^p} + \frac{g(w)^p}{|1 - \bar{z}w|^p} \right].$$

It follows that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z, w)|^p dA_\alpha(z) dA_\alpha(w) \leq 2C \int_{\mathbb{D}} g(z)^p dA_\alpha(z) \int_{\mathbb{D}} \frac{dA_\alpha(w)}{|1 - \bar{z}w|^p}.$$

When $p < 2 + \alpha$, an application of Lemma 10 shows that there exists another constant $C > 0$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z, w)|^p dA_{\alpha}(z) dA_{\alpha}(w) \leq C \int_{\mathbb{D}} g(z)^p dA_{\alpha}(z).$$

This shows that L maps $A_{\alpha}^p(\mathbb{D})$ into $A_{\alpha}^p(\mathbb{D}^2)$. A standard argument based on the closed graph theorem then shows that the operator

$$L : A_{\alpha}^p(\mathbb{D}) \rightarrow A_{\alpha}^p(\mathbb{D}^2)$$

must be bounded.

On the other hand, let us suppose that

$$L : A^p(\mathbb{D}) \rightarrow A_{\alpha}^p(\mathbb{D}^2)$$

is bounded. Then by Lemma 9, the operator $D = \Delta \circ L$ maps $A_{\alpha}^p(\mathbb{D})$ boundedly into $A_{2(\alpha+1)}^p(\mathbb{D})$. It is easy to see that $Df = f'$. Therefore, $f \in A_{\alpha}^p(\mathbb{D})$ would imply that

$$\int_{\mathbb{D}} |(1 - |z|^2)f'(z)|^p dA_{2(\alpha+1)-p}(z) < \infty,$$

which, according to Lemma 5, is a condition that is strictly stronger than $f \in A_{\alpha}^p(\mathbb{D})$ when $p > 2 + \alpha$. So our lifting theorem cannot possibly hold for $p > 2 + \alpha$. \square

We mention that the case $p = 2 + \alpha$ is not covered by the above result. When $\alpha = 0$, we can show by Taylor expansion that the operator L does not map $A^2(\mathbb{D})$ into $A^2(\mathbb{D}^2)$ (these are the unweighted Bergman spaces). In fact, if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is a function in $A^2(\mathbb{D})$, then

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}.$$

On the other hand,

$$L(f)(z, w) = \sum_{k=0}^{\infty} a_k \frac{z^k - w^k}{z - w} = \sum_{k=1}^{\infty} a_k \sum_{i+j=k-1} z^i w^j,$$

and for $k \neq m$, the homogeneous polynomials

$$\sum_{i+j=k-1} z^i w^j, \quad \sum_{i+j=m-1} z^i w^j,$$

are orthogonal with respect to the measure $dA(z)dA(w)$ on \mathbb{D}^2 . So the integral

$$I = \int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z, w)|^2 dA(z) dA(w)$$

can be computed as follows.

$$\begin{aligned} I &= \sum_{k=1}^{\infty} |a_k|^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \left| \sum_{i+j=k-1} z^i w^j \right|^2 dA(z) dA(w) \\ &= \sum_{k=1}^{\infty} |a_k|^2 \sum_{i+j=k-1} \frac{1}{(i+1)(j+1)} \\ &= \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \sum_{i+j=k-1} \left[\frac{1}{i+1} + \frac{1}{j+1} \right] \\ &= 2 \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \sum_{j=0}^{k-1} \frac{1}{j+1} \\ &\sim \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \log(k+1) \\ &\sim \int_{\mathbb{D}} |f(z)|^2 \log \frac{1}{1-|z|^2} dA(z). \end{aligned}$$

This shows that the integral I is not necessarily finite, so the symmetric lifting operator L does not map $A^2(\mathbb{D})$ into $A^2(\mathbb{D}^2)$.

The following result tells us what happens when $p > \alpha + 2$.

Theorem 12. *Suppose $\alpha > -1$, $p > \alpha + 2$, and β is determined by*

$$2(\beta + 1) = p + \alpha.$$

Then the symmetric lifting operator L maps $A_{\alpha}^p(\mathbb{D})$ boundedly into $A_{\beta}^p(\mathbb{D}^2)$.

Proof. Given $f \in A_{\alpha}^p(\mathbb{D})$, we once again appeal to Theorem 6 to obtain a function $g \in L^p(\mathbb{D}, dA_{\alpha})$ such that condition (5) holds. We then have

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z, w)|^p dA_{\beta}(z) dA_{\beta}(w) \leq 2C \int_{\mathbb{D}} g(z)^p dA_{\beta}(z) \int_{\mathbb{D}} \frac{dA_{\beta}(w)}{|1 - \bar{z}w|^p}.$$

Since $p + \alpha = 2(\beta + 1)$ and $p > \alpha + 2$, we must have $\beta > -1$ and $\beta > \alpha$. We write the inner integral above as

$$(\beta + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\beta} dA(w)}{|1 - \bar{z}w|^{2+\beta+(\beta-\alpha)}}$$

and apply Lemma 10 to find another positive constant C such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z, w)|^p dA_\beta(z) dA_\beta(w) \leq C \int_{\mathbb{D}} g(z)^p dA_\alpha(z).$$

This along with the closed graph theorem proves the desired result. \square

Once again, with the help of Lemmas 9 and 5, we can check that the lifting effect of L guaranteed by Theorem 12 is best possible.

5. GENERALIZATION TO THE UNIT BALL

In this section we explain how to generalize our main result to the context of the unit ball in \mathbb{C}^n . Thus we let

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$$

denote the open unit ball in \mathbb{C}^n . For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z),$$

where dv is normalized volume measure on \mathbb{B}_n and c_α is a positive normalizing constant so that $v_\alpha(\mathbb{B}_n) = 1$.

For $p > 0$ and $\alpha > -1$ let

$$A_\alpha^p(\mathbb{B}_n) = L^p(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n)$$

denote the weighted Bergman spaces, where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in \mathbb{B}_n . See [12] for basic properties of these spaces.

For any $a \in \mathbb{B}_n$ there exists a biholomorphic map φ_a on \mathbb{B}_n such that $\varphi_a(0) = a$ and $\varphi_a^{-1} = \varphi_a$. These are sometimes called symmetries (or involutive automorphisms) of \mathbb{B}_n . Explicit formulas are available for φ_a ; see [8] or [12].

It is well known that the Bergman metric on \mathbb{B}_n induces the following distance:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

It follows that

$$\rho(z, w) = |\varphi_z(w)|$$

is also a distance function on \mathbb{B}_n . We shall also call ρ the pseudo-hyperbolic metric on \mathbb{B}_n . The Euclidean metric on \mathbb{B}_n is still denoted by $|z - w|$.

Theorem 13. *Suppose $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

- (a) f belongs to $A_\alpha^p(\mathbb{B}_n)$.
- (b) There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{B}_n .

(c) *There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{B}_n .

(d) *There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that*

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w))$$

for all z and w in \mathbb{B}_n .

The proof follows the same ideas used in the proof of Theorems 6, 7, and 8. Lemma 2 holds for the ball without any change; see [12].

The only change needed in Lemma 3 is the exponent of $1 - |z|^2$. In the context of \mathbb{B}_n , it should be $(1 - |z|^2)^{n+1}$ instead of $(1 - |z|^2)^2$; see [12]. Similarly, the volume of $D(z, r)$ (or $E(z, R)$) is comparable to $(1 - |z|^2)^{n+1}$ whenever r (or R) is fixed.

Lemmas 4 and 5 need to be modified substantially before they can be used. These are the contents of the next two lemmas. But first, we recall three useful notations of differentiation in \mathbb{B}_n .

Given $f \in H(\mathbb{B}_n)$, we write

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z),$$

and call it the radial derivative of f at z . In fact, $Rf(z)$ is the directional derivative of f at z in the radial direction (that is, the direction in z):

$$Rf(z) = \lim_{t \rightarrow 1} \frac{f(tz) - f(z)}{t},$$

where t is a scalar.

The complex gradient of f at z is defined by

$$|\nabla f(z)| = \left[\sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right|^2 \right]^{1/2}.$$

And the invariant complex gradient of f at z is given by

$$|\tilde{\nabla} f(z)| = |\nabla(f \circ \varphi_z)(0)|, \quad z \in \mathbb{B}_n.$$

We can now state the analogs of Lemmas 4 and 5 that will suit our needs.

Lemma 14. *Suppose $z \in \mathbb{B}_n$ and $w = tz$, where t is a scalar. Then*

$$\lim_{w \rightarrow z} \frac{\rho(w, z)}{|z - w|} = \lim_{w \rightarrow z} \frac{\beta(z, w)}{|z - w|} = \frac{1}{1 - |z|^2}.$$

Proof. This follows from the explicit formulas for φ_a given in [8] and [12]. \square

Lemma 15. *Suppose $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

- (a) *The function f is in $A_\alpha^p(\mathbb{B}_n)$.*
- (b) *The function $(1 - |z|^2)Rf(z)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*
- (c) *The function $(1 - |z|^2)|\nabla f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*
- (d) *The function $|\tilde{\nabla} f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*

Proof. See [12]. □

We can now outline the proof of Theorem 13.

First assume that there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$(6) \quad |f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{B}_n . We then fix z in \mathbb{B}_n and let $w = tz$, where t is a scalar. Then

$$\frac{|f(z) - f(w)|}{|z - w|} \leq \frac{\rho(z, w)}{|z - w|}(g(z) + g(w))$$

for all $z \neq w$ in \mathbb{B}_n . Let w approach z in the radial direction and apply Lemma 14. We obtain

$$(1 - |z|^2)|Rf(z)| \leq 2g(z)$$

for all $z \in \mathbb{B}_n$. According to Lemma 15, this is the same as $f \in A_\alpha^p(\mathbb{B}_n)$.

On the other hand, for any holomorphic function f in \mathbb{B}_n and any $z \in \mathbb{B}_n$, we have

$$f(z) - f(0) = \int_0^1 \left[\sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(tz) \right] dt.$$

It follows that for $\rho(z, 0) < r$, where $r \in (0, 1)$ is any fixed radius in the pseudo-hyperbolic metric, we have

$$|f(z) - f(0)| \leq |z| \sup\{|\nabla f(w)| : w \in D(0, r)\}.$$

It is easy to see that in the relatively compact set $D(0, r)$ the Euclidean metric is comparable to the pseudo-hyperbolic metric (as well as the Bergman metric β). It is also easy to see that $|\nabla f(w)|$ is comparable to $|\tilde{\nabla} f(w)|$ in the relatively compact set $D(0, r)$. So we can find a constant $C > 0$, that depends on r but not on f , such that

$$|f(z) - f(0)| \leq C\rho(z, 0) \sup\{|\tilde{\nabla} f(w)| : w \in D(0, r)\}$$

for all $z \in D(0, r)$. Replace f by $f \circ \varphi_w$, then replace z by $\varphi_w(z)$, and use the Möbius invariance of the pseudo-hyperbolic metric and the invariant gradient. We obtain

$$|f(z) - f(w)| \leq C\rho(z, w) \sup\{|\tilde{\nabla} f(u)| : u \in D(z, r)\}$$

for all z and w in \mathbb{B}_n with $\rho(z, w) < r$. Let

$$g(z) = \frac{|f(z)|}{r} + C \sup\{|\tilde{\nabla} f(u)| : u \in D(z, r)\}.$$

Then it is clear that condition (6) is satisfied, and just like in the proof of Theorem 6, $g \in L^p(\mathbb{B}_n, dv_\alpha)$.

So conditions (a) and (b) are equivalent in Theorem 13.

If condition (b) holds in Theorem 13, then condition (c) holds for the same function g , because $\rho \leq \beta$. If condition (c) holds, then an application of Lemma 14 shows that

$$(1 - |z|^2)|Rf(z)| \leq 2g(z)$$

for all $z \in \mathbb{B}_n$, which, according to Lemma 15, implies that condition (a). Therefore, conditions (a), (b), and (c) are all equivalent.

Now let us assume that condition (d) holds, so there exists a continuous function g in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that

$$(7) \quad |f(z) - f(w)| \leq |z - w|(g(z) + g(w))$$

for all z and w in \mathbb{B}_n . Rewrite this as

$$\frac{|f(z) - f(w)|}{|z - w|} \leq g(z) + g(w)$$

and let w approach z in the direction of a complex coordinate axis. We obtain

$$\left| \frac{\partial f}{\partial z_k}(z) \right| \leq 2g(z), \quad 1 \leq k \leq n,$$

so that

$$|\nabla f(z)| \leq 2\sqrt{n}g(z)$$

for all $z \in \mathbb{B}_n$. This together with the assumption that $g \in L^p(\mathbb{B}_n, dv_{p+\alpha})$ shows that the function $(1 - |z|^2)|\nabla f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$. In view of Lemma 15, this is the same as $f \in A_\alpha^p(\mathbb{B}_n)$. So condition (d) implies (a) in Theorem 13.

Finally let us assume that condition (b) holds in Theorem 13. It follows from the well-known identity (see [8] or [12])

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2}$$

that

$$\frac{\rho(z, w)^2}{|z - w|^2} = \frac{|z - w|^2 + |\langle z, w \rangle|^2 - |z|^2|w|^2}{|z - w|^2|1 - \langle z, w \rangle|^2}.$$

By the triangle inequality for the natural inner product in \mathbb{C}^n , we always have

$$|\langle z, w \rangle|^2 \leq |z|^2|w|^2.$$

We deduce that

$$\rho(z, w) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|}$$

for all z and w in \mathbb{B}_n . Just like in the proof of Theorem 8, this along with condition (b) implies the existence of a continuous function g in the space $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that condition (7) holds. So condition (b) implies (d), and the proof of Theorem 13 is complete.

Although we have not checked whether the function

$$d(z, w) = \frac{|z - w|}{|1 - \langle z, w \rangle|}, \quad z, w \in \mathbb{B}_n,$$

defines a metric on \mathbb{B}_n , it is clear by now that the pseudo-hyperbolic metric ρ used in Theorem 13 can be replaced by d . In other words, a holomorphic function f in \mathbb{B}_n belongs to the Bergman space $A_\alpha^p(\mathbb{B}_n)$ if and only if there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq d(z, w)(g(z) + g(w))$$

for all z and w in \mathbb{B}_n .

REFERENCES

- [1] P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, Providence, Rhode Island, 2003.
- [2] P. Duren and A. Shields, Restriction of H^p functions to the diagonal of the polydisk, *Duke Math. J.* **42** (1975), 751-753.
- [3] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [4] P. Hajlasz, Sobolev spaces on an arbitrary metric space, *Potential Anal.* **5** (1996), 403-415.
- [5] P. Hajlasz, Sobolev spaces on metric-measure spaces, *Contemp. Math.* **338** (2003), 173-218.
- [6] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, 2000.
- [7] C. Horowitz and D. Oberlin, Restriction of $H^p(\mathbb{D}^n)$ functions to the diagonal of \mathbb{D}^n , *Indiana Univ. Math. J.* **24** (1974/75), 767-772.
- [8] W. Rudin, *Function Theory in the Unit Ball of \mathbb{C}^n* , Springer, New York, 1980.
- [9] E. Saksman, Pointwise characterizations of Hardy-Sobolev spaces, talk at the conference *Holomorphic Function Spaces and Their Operators*, Marseille, June, 2006.
- [10] F. Shamoyan, Applications of Dzhrbashyan integral representations to some problems of analysis, *Dokl. Akad. Nauk SSSR* **261** (1981), 557-561.
- [11] M. Stessin and K. Zhu, Composition operators on embedded disks, to appear in *J. Operator Theory*.
- [12] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2005.

DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, GUANGDONG, CHINA
E-mail address: wulan@stu.edu.cn

DEPARTMENT OF MATHEMATICS, SUNY, ALBANY, NY 12222, USA, AND DE-
PARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, GUANGDONG, CHINA
E-mail address: kzhu@math.albany.edu