THE MÖBIUS INVARIANCE OF BESOV SPACES ON THE UNIT BALL OF $\mathbb{C}^n$

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ABSTRACT. It is well known that, for $1 \le p < \infty$, the diagonal Besov space B_p of the open unit ball admits a norm or semi-norm $\| \, \|_p$ such that $\|f \circ \varphi\|_p = \|f\|_p$ for all f in B_p and all automorphisms φ of the unit ball. We show here that the same result holds when $0 < p < 1$.

1. INTRODUCTION

Throughout the paper we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional complex Euclidean space. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots z_n\overline{w}_n$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} = \sqrt{\langle z, z\rangle}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

We use $\mathrm{Aut}(\mathbb{B}_n)$ to denote the group of all automorphisms, or biholomorphic maps, of $\mathbb{B}_n$.

Suppose $0 < p < \infty$ and N is a positive integer such that $Np > n$. The (diagonal) Besov space B_p consists of holomorphic functions f in $\mathbb{B}_n$ such that the functions

$$(1 - |z|^2)^N \partial^\alpha f(z), \quad |\alpha| = N,$$

all belong to $L^p(\mathbb{B}_n, d\lambda)$, where

$$\alpha = (\alpha_1, \cdots, \alpha_n)$$

is a multi-index of nonnegative integers,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

is the size of α,

$$\partial^\alpha f(z) = \frac{\partial^\alpha f}{\partial z^\alpha}(z) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(z)$$

is a mixed partial derivative, and

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the Möbius invariant measure on $\mathbb{B}_n$. Here we use dv to denote volume measure on $\mathbb{B}_n$ normalized so that that $v(B_n) = 1$.

It is well known that the above definition of B_p is independent of the choice of N; see Section 6.1 of [3]. In what follows we fix a positive integer N such that $Np > n$ and define a "norm" on B_p as follows:

$$\|f\|_p = \sum_{|\alpha|<N} \left|\frac{\partial^\alpha f}{\partial z^\alpha}(0)\right| + \sum_{|\alpha|=N} \left[\int_{\mathbb{B}_n} \left|(1-|z|^2)^N \partial^\alpha f(z)\right|^p \, d\lambda(z)\right]^{1/p}.$$

Of course, when $0 < p < 1$, $\|f\|_p$ is not really a norm; B_p is a complete metric space in this case.

The main result of the paper is the following.

Theorem 1. *For any* $0 < p \le 1$ *there exists a positive constant* $C > 0$ *such that* $\|f \circ \varphi\|_p \le C\|f\|_p$ *for all* $f \in B_p$ *and* $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

The constant C above depends on p, n, and N, but is independent of f and φ. It follows that

$$\|f\|_{p'} = \sup\{\|f \circ \varphi\|_p : \varphi \in \mathrm{Aut}(\mathbb{B}_n)\}$$

defines a Möbius invariant "norm" on B_p, $0 < p \le 1$, in the sense that $\|f \circ \varphi\|_{p'} = \|f\|_{p'}$ for all $f \in B_p$ and $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

When $1 < p < \infty$, there exist unbounded functions in B_p. Therefore, the above theorem cannot possibly hold for B_p when $p > 1$ (because of the presence of the term $|f(0)|$ in the definition of $\|f\|_p$). However, every B_p, $p > 1$, admits a Möbius invariant semi-norm; see [1] and [3].

2. THE AUTOMORPHISM GROUP

In this section we gather some information concerning the automorphism group $\mathrm{Aut}(\mathbb{B}_n)$ that is necessary for the proof of the main theorem.

Every unitary transformation U of $\mathbb{C}^n$, or equivalently, every unitary matrix U, is clearly an automorphism of $\mathbb{B}_n$. It is well known that the volume measure dv is invariant under the action of unitary transformations, that is,

$$\int_{\mathbb{B}_n} f(Uz)\, dv(z) = \int_{\mathbb{B}_n} f(z)\, dv(z) \tag{1}$$

whenever $f \in L^1(\mathbb{B}_n, dv)$.

Lemma 2. *Suppose $0 < p \le 1$ and α' is a multi-index of nonnegative integers with size N. Then*

$$|\partial^{\alpha'}(f(Uz))|^p \le n^N \sum_{|\alpha|=N} |(\partial^\alpha f)(Uz)|^p$$

for all f holomorphic in $\mathbb{B}_n$ and all unitary transformations U in $\mathrm{Aut}(\mathbb{B}_n)$.

Proof. For a unitary matrix $U = (u_{ij})$ and holomorphic f let $g(z) = f(Uz)$. Then by the chain rule, we have

$$\frac{\partial g}{\partial z_{n_1}}(z) = \sum_{j_1=1}^n \frac{\partial f}{\partial z_{j_1}}(Uz)u_{j_1,n_1},$$

and

$$\frac{\partial^2 g}{\partial z_{n_1}\partial z_{n_2}}(z) = \sum_{j_1=1}^n \sum_{j_2=1}^n \frac{\partial^2 f}{\partial z_{j_1} z_{j_2}}(Uz)u_{j_1,n_1}u_{j_2,n_2}.$$

More generally,

$$\frac{\partial^N g}{\partial z_{n_1}\cdots\partial z_{n_N}}(z) = \sum_{j_1}^n \cdots \sum_{j_N=1}^n \frac{\partial^N f}{\partial z_{j_1}\cdots\partial z_{j_N}}(Uz)u_{j_1,n_1}\cdots u_{j_N,n_N}.$$

Since each entry u_{ij} of a unitary matrix has modulus less than or equal to 1, and since

$$\left|\frac{\partial^N f}{\partial z_{j_1}\cdots\partial z_{j_N}}(Uz)\right|^p \le \sum_{|\alpha|=N} |(\partial^\alpha f)(Uz)|^p$$

for any choice of $j_1, j_2, \cdots, j_N$, the desired result is then a consequence of Hölder's inequality. □

Combining Lemma 2 and (1), we obtain the following.

Lemma 3. *If $0 < p \le 1$, f is holomorphic in $\mathbb{B}_n$, and U is a unitary transformation, then $\|f \circ U\|_p \le C\|f\|_p$, where C is a positive constant independent of f and U.*

Another class of automorphisms, called involutions, can be defined explicitly. Thus for every point $a \in \mathbb{B}_n$, $a \ne 0$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}\,Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n,$$

where

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2}a, \qquad z \in \mathbb{C}^n,$$

is the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by a, and $Q_a(z) = z - P_a(z)$ is the orthogonal projection from $\mathbb{C}^n$ onto $[a]^\perp$. When $a = 0$, we simply define $\varphi_a(z) = -z$.

It is well known that each φ_a is an automorphism of $\mathbb{B}_n$. Furthermore, φ_a has the properties that

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \varphi_a \circ \varphi_a(z) = z, \tag{2}$$

and

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1-|a|^2)(1-\langle z, w\rangle)}{(1-\langle z, a\rangle)(1-\langle a, w\rangle)}, \tag{3}$$

where z and w are arbitrary points in $\mathbb{B}_n$. See [2] or [3].

If φ is any automorphism of $\mathbb{B}_n$ and $a = \varphi(0)$, then the automorphism $\varphi_a \circ \varphi$ fixes the origin and hence must be a unitary transformation (by a classical theorem of Cartan; see [2] or [3]). This shows that $\varphi = \varphi_a \circ U$ for some $a \in \mathbb{B}_n$ and some unitary U. A similar argument shows that every $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ can be written as $V \circ \varphi_b$ for some $b \in \mathbb{B}_n$ and some unitary V. We summarize this analysis as the following.

Lemma 4. *Every $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ can be written as*

$$\varphi = \varphi_a \circ U = V \circ \varphi_b,$$

where U and V are unitary transformations, $a = \varphi(0)$, and $\varphi(b) = 0$.

The following result gives an even more detailed decomposition for automorphisms.

Lemma 5. *Every automorphism φ of $\mathbb{B}_n$ can be decomposed into the form*

$$\varphi = U \circ \varphi_a \circ V,$$

where U and V are unitary transformations, and $a = (r, 0, \cdots, 0)$ for some $r \in [0, 1)$.

Proof. Fix an automorphism $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$ and choose a unitary transformation U such that $U(a) = \varphi(0)$, where $a = (r, 0, \cdots, 0)$ and $r = |\varphi(0)|$. This is possible, because the unitary group is transitive on any sphere centered at the origin. Since

$$\varphi_a \circ U^{-1} \circ \varphi(0) = 0,$$

an application of Lemma 4 shows that there exists a unitary transformation V such that

$$\varphi_a \circ U^{-1} \circ \varphi = V,$$

or

$$\varphi = U \circ \varphi_a \circ V,$$

completing the proof of the lemma. $\square$

3. MÖBIUS INVARIANCE OF B_p

Let M_n be the dimensional constant defined by $M_n = 1$ for $n = 1$ and $M_n = 2n$ for $n > 1$. The Möbius invariance of B_p exhibits a subtle difference when p crosses this constant.

When $p > M_n$, the Besov space B_p can be given an explicit Möbius invariant semi-norm:

$$\left[\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \, d\lambda(z)\right]^{1/p},$$

where $|\widetilde{\nabla} f(z)|$ is the Möbius invariant complex gradient of f at z. This integral diverges for $p \le M_n$ unless f is constant. See Section 6.3 of [3].

When $p = 2$, the Dirichlet type space B_2 admits the following Möbius invariant semi-inner product:

$$\langle f, g\rangle = \sum_{\alpha} |\alpha| \frac{\alpha!}{|\alpha|!} a_\alpha \bar{b}_\alpha,$$

where a_α and b_α are the Taylor coefficients of f and g, respectively, and $\alpha! = \alpha_1! \cdots \alpha_n!$. See Section 6.4 of [3].

When $p = 1$, the space B_1 consists exactly of functions of the form

$$f(z) = c_0 + \sum_{k=1}^{\infty} c_k f_k(z), \tag{4}$$

where $\{c_k\} \in l^1$ and f_k are coordinate functions of automorphisms. Moreover,

$$\|f\|_m = \inf \sum_{k=0}^{\infty} |c_k|,$$

where the infimum is taken over all $\{c_k\}$ satisfying (4), defines a Möbius invariant norm (not just a semi-norm) on B_1. See Section 6.2 of [3].

More generally, for any $p \in (1, \infty)$, the Möbius invariance of B_p follows from complex interpolation between B_1 and B_q, where $q > \max(p, M_n)$. See [1] or [3].

We proceed to show that B_p is also Möbius invariant when $0 < p < 1$. The following integral estimate will be essential for us.

Lemma 6. *For $s > -1$ and t real the integral*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s \, dv(w)}{|1 - \langle z, w\rangle|^{n+1+s+t}}$$

has the following asymptotic properties as $|z| \to 1^-$.

(a) *If $t > 0$, then $I(z)$ is bounded.*
(b) *If $t = 0$, then $I(z)$ is comparable to $-\log(1 - |z|^2)$.*

(c) *If* $t > 0$*, then* $I(z)$ *is comparable to* $(1-|z|^2)^{-t}$*.*

Proof. See Proposition 1.4.10 of [2]. □

The key ingredient in our proof is the following atomic decomposition for functions in Besov spaces.

Lemma 7. *For any* $0 < p < \infty$ *and* $b > \max(0, n(p-1)/p)$ *there exists a sequence* $\{a_k\}$ *in* $\mathbb{B}_n$ *such that* B_p *consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{\infty} c_k \left(\frac{1-|a_k|^2}{1-\langle z, a_k\rangle}\right)^b, \tag{5}$$

where $\{c_k\} \in l^p$*. Moreover,* $\|f\|_p^p$ *is equivalent to*

$$\inf \sum_{k=1}^{\infty} |c_k|^p,$$

where the infimum is taken over all sequences $\{c_k\}$ *satisfying (5).*

Proof. See Section 6.1 of [3]. □

Let H^∞ be the space of bounded holomorphic functions in $\mathbb{B}_n$ equipped with the sup-norm $\|f\|_\infty$.

Lemma 8. *If* $0 < p \le 1$*, then* B_p *is continuously contained in* H^∞*.*

Proof. If $0 < p \le 1$, then Hölder's inequality shows that l^p is continuously contained in l^1. Since each function

$$f_k(z) = \left(\frac{1-|a_k|^2}{1-\langle z, a_k\rangle}\right)^b$$

has sup-norm less than or equal to 2^b, the desired result then follows from Lemma 7. □

The above proof actually shows that B_p, when $0 < p \le 1$, is continuously contained in the disk algebra.

We can now prove Theorem 1, the main result of the paper.

Fix a function $f \in B_p$, where $0 < p \le 1$. By Lemma 8, there exists a constant $C_1 > 0$ such that

$$\sup\{|f\circ\varphi(z)| : z \in \mathbb{B}_n, \varphi \in \mathrm{Aut}(\mathbb{B}_n)\} = \|f\|_\infty \le C_1\|f\|_p.$$

Combining this with Cauchy's estimates, we can find a constant $C_2 > 0$ such that

$$\sum_{|\alpha|<N} |\partial^m(f\circ\varphi)(0)|^p \le C_2\|f\|_p \tag{6}$$

for all $\varphi \in \mathrm{Aut}(\mathbb{B}_n)$.

By Lemma 7 (with $b = 1$), there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ and a sequence $\{c_k\}$ in l^p such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{1-|a_k|^2}{1-\langle z, a_k\rangle},$$

where

$$\sum_{k=1}^{\infty} |c_k|^p \le C_3 \|f\|_p^p, \tag{7}$$

and C_3 is a positive constant independent of f and $\{c_k\}$. Since $0 < p \le 1$, it follows from Hölder's inequality that the integral

$$\int_{\mathbb{B}_n} \left|(1-|z|^2)^N \partial^\alpha (f\circ\varphi)(z)\right|^p \, d\lambda(z)$$

is less than or equal to

$$\sum_{k=1}^{\infty} |c_k|^p \int_{\mathbb{B}_n} \left|(1-|z|^2)^N \partial^\alpha (f_k\circ\varphi)(z)\right|^p \, d\lambda(z),$$

where α is a multi-index with size N and

$$f_k(z) = \frac{1-|a_k|^2}{1-\langle z, a_k\rangle} = 1 - \langle \varphi_{a_k}(z), a_k\rangle, \qquad z \in \mathbb{B}_n.$$

Note that the last equality above follows from (3).

By Lemma 4, we have $\varphi_{a_k}\circ\varphi = U_k \circ \varphi_{a'_k}$, where U_k is a unitary transformation and $a'_k = \varphi^{-1}(a_k)$. It follows that

$$f_k \circ \varphi(z) = 1 - \langle \varphi_{a'_k}(z), b'_k\rangle,$$

where $b'_k = U_k^*(a_k)$.

For u and v in $\mathbb{B}_n$ we consider the function

$$f_{u,v}(z) = 1 - \langle \varphi_u(z), v\rangle, \qquad z \in \mathbb{B}_n.$$

The proof of the theorem will be complete if we can show that there exists a constant $C_4 > 0$, independent of u and v, such that

$$\int_{\mathbb{B}_n} \left|(1-|z|^2)^N \partial^\alpha f_{u,v}(z)\right|^p \, d\lambda(z) \le C_4 \tag{8}$$

for all u and v in $\mathbb{B}_n$ and all α with $|\alpha| = N$.

Apply Lemma 5 to the automorphism φ_u and then use Lemma 3. We conclude that it suffices to prove (8) under the additional assumption that $u = (r, 0, \cdots, 0)$, where $0 \le r < 1$. In this case, we have

$$\varphi_u(z) = \left(\frac{r - z_1}{1 - r z_1}, -\frac{\sqrt{1-r^2}}{1-rz_1} z_2, \cdots, -\frac{\sqrt{1-r^2}}{1-rz_1} z_n\right),$$

and so

$$f_{u,v}(z)=1-\overline{v}_1\frac{r-z_1}{1-rz_1}+\overline{v}_2\frac{\sqrt{1-r^2}}{1-rz_1}z_2+\cdots+\overline{v}_n\frac{\sqrt{1-r^2}}{1-rz_1}z_n,$$

where $(v_1,\cdots,v_n)=v$.

Consider a mixed partial derivative $\partial^\alpha(f_{u,v})(z)$, where $|\alpha|=N$. It is clear that this partial derivative is zero when

$$|\alpha_2|+\cdots+|\alpha_n|>1.$$

Therefore, we can assume that at most one of the numbers in $\{\alpha_2,\cdots,\alpha_n\}$ is 1 and the rest are all 0. There are two cases to consider.

The first case is when $\alpha_2=\cdots=\alpha_n=0$. Then it is easy to check that

$$\partial^\alpha(f_{u,v})(z)=\frac{\partial^N f_{u,v}}{\partial z_1^N}(z)=\frac{N!r^{N-1}}{(1-rz_1)^{N+1}}\langle w,v\rangle,$$

where

$$w=(1-r^2,r\sqrt{1-r^2}\,z_2,\cdots,r\sqrt{1-r^2}\,z_n).$$

Since

$$|w|^2=(1-r^2)^2+r^2(1-r^2)(|z|^2-|z_1|^2)\le(1-r^2)^2+(1-r^2)(1-|z_1|^2),$$

an application of the Cauchy-Schwarz inequality followed by Hölder's inequality gives

$$|\partial^\alpha(f_{u,v})(z)|\le N!\frac{1-r^2+\sqrt{1-r^2}\sqrt{1-|z_1|^2}}{|1-rz_1|^{N+1}}. \tag{9}$$

Writing rz_1 as the inner product of z with $(r,0,\cdots,0)$ and applying part (c) of Lemma 6, we see that

$$\int_{\mathbb{B}_n}\left(\frac{(1-r^2)(1-|z|^2)^N}{|1-rz_1|^{N+1}}\right)^p d\lambda(z)$$

is a bounded function of r. A similar estimate works for the second term on the right hand side of (9), after we make use of the fact that

$$1-|z_1|^2\le 2|1-rz_1|.$$

This proves the inequality (8) when $\alpha_k=0$ for $2\le k\le n$.

The second case is when exactly one of the integers in $\{\alpha_2,\cdots,\alpha_n\}$, say α_k, is 1 and the rest are all 0. In this case, we have

$$\partial^\alpha(f_{u,v})(z)=(N-1)!\overline{v}_k r^{N-1}\frac{\sqrt{1-r^2}}{(1-rz_1)^N}.$$

By part (b) of Lemma 6, the integral

$$\int_{\mathbb{B}_n}\left(\frac{\sqrt{1-r^2}}{|1-rz_1|^N}(1-|z|^2)^N\right)^p d\lambda(z)$$

is dominated by

$$(1-r^2)^{p/2}\log\frac{2}{1-r^2},$$

which is clearly a bounded function of $r \in [0,1)$.

This completes the proof of Theorem 1.

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