HOLOMORPHIC MEAN LIPSCHITZ SPACES AND HARDY SOBOLEV SPACES ON THE UNIT BALL

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ABSTRACT. We study two classes of holomorphic functions in the unit ball $B^n$ of $\mathbb{C}^n$: mean Lipschitz spaces and Hardy Sobolev spaces. Main results include new characterizations in terms of fractional radial differential operators and various comparisons between these two classes.

1. INTRODUCTION

For points $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w \rangle = z_1w_1 + \cdots + z_nw_n, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Let $B^n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball and let $S^n = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ denote the unit sphere in $\mathbb{C}^n$. The normalized Lebesgue measures on $B^n$ and $S^n$ will be denoted by $dv$ and $d\sigma$, respectively.

Let $H(B^n)$ denote the space of all holomorphic functions in $B^n$. Given $0 < r < 1$, $0 < p < \infty$, and $f \in H(B^n)$, we define

$$M_p(r, f) = \left[ \int_{S^n} |f(r\zeta)|^p \, d\sigma(\zeta) \right]^{1/p}$$

and call it the $L^p$-mean of $f$ over the sphere $|z| = r$. When $p = \infty$, we write

$$M_\infty(r, f) = \sup \{|f(r\zeta)| : \zeta \in S^n\}.$$

For $0 < p \leq \infty$ the Hardy space $H^p$ consists of all functions $f \in H(B^n)$ such that

$$\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty.$$

See [20] for basic information about the Hardy spaces.

For any real $t$ we consider the special fractional radial differential operator $R^t$ defined on $H(B^n)$ by

$$R^t f(z) = f(0) + \sum_{k=1}^\infty k^t f_k(z),$$

where

$$f(z) = \sum_{k=0}^\infty f_k(z).$$

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is the homogeneous expansion of $f$. It is easy to see that if $k$ is a positive integer, then $R^k$, modulo the value at the origin, is exactly the $k$-th power of the usual radial derivative $R$ on $H(\mathbb{B}^n)$ defined by

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z).$$

It is also clear that each $R^t$ is invertible on $H(\mathbb{B}^n)$, with $(R^t)^{-1} = R^{-t}$ on $H(\mathbb{B}^n)$.

For $0 < p \leq \infty$ and $\alpha$ real, the Hardy-Sobolev space $H^p_\alpha$ consists of $f \in H(\mathbb{B}^n)$ such that $R^\alpha f \in H^p$. For $f \in H^p_\alpha$ we are going to write

$$\|f\|_{H^p_\alpha} = \|R^\alpha f\|_{H^p}.$$  

See [1] for basic information about the Hardy-Sobolev spaces.

Given $0 < p \leq \infty$, $0 < q \leq \infty$, and $t > -1$, the mixed-norm space $H^{p,q}_t$ consists of all $f \in H(\mathbb{B}^n)$ such that

$$M^p_r(r, f) \in L^q((0,1), (1-r)^t \, dr).$$

These spaces have been studied by numerous authors for a long time. Early references include [4, 8], and more recent work can be found in [10, 11, 12, 13, 15, 18].

In this paper we will be interested in a special class of mixed-norm spaces. More specifically, for $0 < \alpha < \infty$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$, we consider the mixed-norm space $\Lambda^{p,q}_\alpha$ consisting of $f \in H(\mathbb{B}^n)$ such that

$$(1-r)^{k-\alpha} M^p_r(r, R^k f) \in L^q((0,1), \frac{dr}{1-r}),$$

where $k$ is any integer greater than $\alpha$. It can be shown that the resulting space $\Lambda^{p,q}_\alpha$ is independent of the choice of $k$; see [6]. When $q = \infty$, we will write $\Lambda^p_\alpha$ instead of $\Lambda^{p,\infty}_\alpha$.

When $p = q = \infty$, the space $\Lambda^\infty_\alpha$ consists of $f \in H(\mathbb{B}^n)$ such that

$$\sup_{z \in \mathbb{B}^n} (1-|z|^2)^{k-\alpha}|R^k f(z)| < \infty,$$

where $k$ is any integer greater than $\alpha$. This is exactly the classical holomorphic Lipschitz space $\Lambda_\alpha$; see [20]. On the other hand, it was shown in [6] that for $0 < \alpha < 1$ the space $\Lambda^{p,q}_\alpha$ consists of functions $f \in H(\mathbb{B}^n)$ such that

$$\int_0^1 \omega^\alpha_p(t, f) \frac{dt}{t^{1+\alpha q}} < \infty,$$

where

$$\omega_p(t, f) = \sup_{\|U-I\| \leq t} \left[ \int_{\mathbb{S}^n} |f(U\zeta) - f(\zeta)|^p \, d\sigma(\zeta) \right]^{1/p}.$$

Here $U$ denotes a unitary matrix acting on $\mathbb{C}^n$ and

$$\|U - I\| = \sup \{|U\zeta - \zeta : \zeta \in \mathbb{S}^n\}.$$  

This motivates us to call the spaces $\Lambda^{p,q}_\alpha$ holomorphic mean Lipschitz spaces. See [6] for recent research on these spaces.

The purpose of this paper is to obtain new characterizations for the spaces $H^p_\alpha$ and $\Lambda^{p,q}_\alpha$ and to make various comparisons between the two classes.
2. Preliminaries on Fractional Derivatives

In addition to the one-parameter fractional radial differential operators $R^s$ defined in the introduction, we will also make extensive use of a two-parameter fractional radial differential operator $R^{s,t}$ on $H(\mathbb{B}^n)$. More specifically, we consider real parameters $s$ and $t$ (although the following definition makes sense for complex parameters as well, we will only need the real case) with the property that neither $n + s$ nor $n + s + t$ is a negative integer. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of $f \in H(\mathbb{B}^n)$, we define

$$R^{s,t} f(z) = \sum_k \frac{\Gamma(n + 1 + s)\Gamma(n + 1 + k + s + t)}{\Gamma(n + 1 + s + t)\Gamma(n + 1 + k + s)} f_k(z),$$

and

$$R_{s,t} f(z) = \sum_k \frac{\Gamma(n + 1 + s + t)\Gamma(n + 1 + k + s)}{\Gamma(n + 1 + s)\Gamma(n + 1 + k + s + t)} f_k(z).$$

When $t > 0$, an application of the Stirling’s formula shows that

$$\frac{\Gamma(n + 1 + s)\Gamma(n + 1 + k + s + t)}{\Gamma(n + 1 + s + t)\Gamma(n + 1 + k + s)} \sim k^t \quad \text{as} \quad k \to \infty.$$ 

Therefore, $R^{s,t}$ is indeed a fractional radial differential operator of order $t$. It is obvious that $R_{s,t}$ is the inverse of $R^{s,t}$ and so will be called a fractional integral operator of order $t$.

One of the main advantages of the operators $R^{s,t}$ and $R_{s,t}$ is that they interact well with Bergman type and Cauchy type kernels on the unit ball. This is made precise by the following result.

**Lemma 2.1** ([20]). Suppose neither $n + s$ nor $n + s + t$ is a negative integer. Then

$$R^{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}},$$

and

$$R_{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}} = \frac{1}{(1 - \langle z, w \rangle)^{n+1+s+t}}.$$

For $s > -1$ the weighted Lebesgue measure $dv_s$ is defined by

$$dv_s(z) = c_s(1 - |z|^2)^s dv(z),$$

where

$$c_s = \frac{\Gamma(n + s + 1)}{n! \Gamma(s + 1)}$$

is a normalizing constant so that $dv_s$ is a probability measure on $\mathbb{B}^n$.

The following lemma is a consequence of the above formulas.

**Lemma 2.2.** Suppose $f$ is holomorphic in $\mathbb{B}^n$. If $s > -1$ and $n + s + t$ is not a negative integer, then

$$R^{s,t} f(z) = \lim_{r \to 1^-} \int_{\mathbb{B}^n} \frac{f(rw)dv_s(w)}{(1 - \langle z, w \rangle)^{n+1+s+t}}.$$
If \( s = -1 \) and \( n + t - 1 \) is not a negative integer, then
\[
R^{-1,t}f(z) = \lim_{r \to 1^-} \int_{S^n} \frac{f(r\zeta)d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+t}}.
\]
Moreover, the limits above always exist.

Proof. The case \( s > -1 \) is proved in [20] and the proof is based on the reproducing property of the Bergman kernel. The case \( s = -1 \) can be proved in a similar fashion using the Cauchy integral formula for the unit ball. \( \square \)

By the definition of the two-parameter fractional radial differential operator \( R^{s,t} \), we have the following elementary relations.

Lemma 2.3. Suppose that neither \( n + s \) nor \( n + s + t \) is a negative integer. Then we have
\[
R^{s,t} = R^{s+1,-t}.
\]

Lemma 2.4. Suppose that none of \( n + \lambda, n + \lambda + t, \) and \( n + \lambda + s + t \) is a negative integer. Then we have
\[
R^{\lambda,t}R^{\lambda+s} = R^{\lambda+s,t}R^{\lambda,s} = R^{\lambda,s+t}.
\]

Lemma 2.5. Suppose \( f \in H(B^n) \) and \( n + s \) is not a negative integer. Then
\[
R^{s,1}f = f + \frac{\mathcal{R}f}{n + s + 1}.
\]

Proof. Differentiating with respect to \( \overline{w} \) at \( w = 0 \) then leads to the result. \( \square \)

Lemma 2.6. Let \( m \) be a positive integer. If \( n + s \) is not a negative integer, then there exist constants \( \{c_0, c_1, \ldots, c_m\} \) such that
\[
R^{s,m} = \sum_{j=0}^{m} c_j R^j,
\]
where \( c_0 = 1 \) and \( c_m \neq 0 \).

Proof. Since
\[
R^{s+j,m-j} = R^{s+j,1}R^{s+j+1,m-j-1}, \quad j = 0, 1, \ldots, m - 2,
\]
we have
\[
R^{s,m} = R^{s,1}R^{s+1,1} \cdots R^{s+m-1,1}.
\]
By Lemma 2.5, we have
\[
R^{s,m} = \left(1 + \frac{\mathcal{R}}{n + s + 1}\right) \left(1 + \frac{\mathcal{R}}{n + s + 2}\right) \cdots \left(1 + \frac{\mathcal{R}}{n + s + m}\right) = \sum_{j=0}^{m} c_j R^j.
\]
\( \square \)
3. Holomorphic Mean Lipschitz Spaces

Note that our definition of the holomorphic mean Lipschitz spaces \( \Lambda^p_{\alpha} \) makes perfect sense when \( \alpha \leq 0 \). The restriction \( \alpha > 0 \) ensures that the functions we consider will have boundary values.

**Lemma 3.1.** Suppose \( 0 < \alpha < \infty, 1 \leq p < \infty, \) and \( 1 \leq q \leq \infty \). Then \( \Lambda^p_{\alpha} \subset H^p \).

**Proof.** Let \( 1/2 < r < 1 \) and \( \sigma_{\alpha} = \alpha - m \), where \( m = [\alpha] \) is the integer part of \( \alpha \). By the fundamental theorem of calculus and Minkowski’s inequality, we have

\[
M_p(r, f) \lesssim \sup_{|z| < 1/2} |f(z)| + \int_{0}^{r} M_p(t, Rf) dt.
\]

Applying this repeatedly and using Fubini’s theorem, we obtain

\[
M_p(r, f) \lesssim \sup_{|z| < 1/2} |f(z)| + \int_{0}^{r} (r-t)^m M_p(t, R^{m+1}f) dt
\]

\[
\lesssim \sup_{|z| < 1/2} |f(z)| + \int_{0}^{1} (1-t)^m M_p(t, R^{m+1}f) dt. \tag{3.1}
\]

On the other hand,

\[
M_p(t, R^{m+1}f)(1-t)^{1-\sigma_{\alpha}} \leq M_p(t, R^{m+1}f) \left( \int_{t}^{1} (1-\rho)^{(1-\sigma_{\alpha})q-1} d\rho \right)^{1/q}
\]

\[
\leq \left( \int_{0}^{1} M_p^q(\rho, R^{m+1}f)(1-\rho)^{(1-\sigma_{\alpha})q-1} d\rho \right)^{1/q} \lesssim 1.
\]

Thus we have

\[
M_p(r, f) \lesssim \sup_{|z| < 1/2} |f(z)| + \int_{0}^{1} (1-t)^{-1+\alpha} dt \lesssim 1
\]

whenever \( \alpha > 0 \). \[\square\]

Recall from [6] that if \( 0 < \alpha < 1, 1 \leq p < \infty, 1 \leq q \leq \infty \), and \( f \in H^p \), then \( f \in \Lambda^p_{\alpha} \) if and only if

\[
\int_{0}^{1} \omega_p^q(t, f) \frac{dt}{t^{1+\alpha q}} < \infty.
\]

This justifies our usage of the term holomorphic mean Lipschitz space for \( \Lambda^p_{\alpha} \) when \( 0 < \alpha < 1 \). Our first objective here is to demonstrate that a similar interpretation can be given for any \( \alpha > 0 \).

We will cover the case \( 0 < \alpha < 2 \) in detail. After that it will be clear how to generalize the result to other ranges of \( \alpha \). The classical approach for the ordinary Lipschitz spaces is well known. More specifically, the Lipschitz space \( \Lambda_{\alpha} \) can be described by the first order difference \( |f(z) - f(w)| \) when \( 0 < \alpha < 1 \), and it can be described by the second order difference \( |f(z + h) + f(z - h) - 2f(z)| \) when \( 0 < \alpha < 2 \), and so on. The situation for mean Lipschitz spaces will be similar.

Thus we consider the second order \( H^p \) mean variation defined as follows:

\[
\omega^2_p(t, f) = \sup_{\|U - l\| \leq t} \left( \int_{\mathbb{R}^n} |f(U\zeta) - 2f(\zeta) + f(U^{-1}\zeta)|^p d\sigma(\zeta) \right)^{1/p}.
\]

We need a few lemmas to describe the spaces \( \Lambda^p_{\alpha} \) in terms of the second order \( H^p \) mean variation.
Lemma 3.2 (Hardy’s inequalities, [16]). Let $h$ be a non-negative function and $1 \leq p < \infty$, $r > 0$. Then we have

\[
\begin{align*}
(i) & \quad \left[ \int_0^1 \left( \int_0^x h(y)dy \right)^p x^{-r-1}dx \right]^{1/p} \leq \frac{p}{r} \left( \int_0^1 (yh(y))^{p+y-r-1}dy \right)^{1/p}; \\
(ii) & \quad \left[ \int_0^1 \left( \int_x^1 h(y)dy \right)^p x^{-r-1}dx \right]^{1/p} \leq \frac{p}{r} \left( \int_0^1 (yh(y))^{p+y-r-1}dy \right)^{1/p}.
\end{align*}
\]

Recall that the Poisson kernel on the unit disk $\mathbb{D}$ is given by

\[ P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}. \]

Lemma 3.3 ([6]). Let $1 \leq p < \infty$ and $0 < r < 1$. Then for $f \in H^p$ we have

\[ M_p(r, R^2f) \lesssim \|f\|_{H^p} + \frac{1}{1 - r} \int_0^\pi P(r, t) \omega^*_p(t, f) \frac{dt}{t}. \]

Lemma 3.4 ([6]). Let $1 \leq p < \infty$ and $0 < t < 1/2$. Then we have

\[ \omega^*_p(t, f) \lesssim t^2 M_p(1 - t, R^2f) + \int_0^t \tau M_p(1 - \tau, R^2f) d\tau. \]

We can now state and prove our first result.

**Theorem 3.5.** Let $0 < \alpha < 2$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $f \in H^p$. Then $f \in \Lambda^{p, q}_\alpha$ if and only if

\[ \int_0^1 \left( \omega^*_p(t, f) \right)^q \frac{1}{t^{1+\alpha q}} dt < \infty. \quad (3.2) \]

**Proof.** First assume that the condition in (3.2) holds. By Lemma 3.3, we have

\[ M_p(r, R^2f) \lesssim \|f\|_{H^p} + \frac{1}{1 - r} \int_0^\pi P(r, t) \omega^*_p(t, f) \frac{dt}{t}. \]

Thus

\[
\int_0^1 (1 - r)^{(2-\alpha)q-1} M^q_p(r, R^2f) dr \\
\lesssim \|f\|^q_{H^p} + \int_0^1 (1 - r)^{(1-\alpha)q-1} \left( \int_0^\pi P(r, t) \omega^*_p(t, f) \frac{dt}{t} \right)^q dr.
\]

From the definition of the Poisson kernel, it is easy to see that

\[ P(r, t) \lesssim \frac{1 - r}{(1 - r)^2 + |t|^2}. \]

By Hardy’s inequality, we have

\[
\int_0^1 (1 - r)^{(1-\alpha)q-1} \left( \int_0^{1-r} P(r, t) \omega^*_p(t, f) \frac{dt}{t} \right)^q dr \\
\lesssim \int_0^1 \left( \frac{1}{\omega^*_p(t, f)} \frac{1}{t^{1+\alpha q}} \right)^q dr \\
\lesssim \int_0^1 \left( \omega^*_p(t, f) \right)^q \frac{1}{t^{1+\alpha q}} dt.
\]
if \( \alpha > 0 \), and another application of Hardy’s inequality shows that
\[
\int_0^1 (1 - r)^{(1 - \alpha)q - 1} \left( \int_{1-r}^r P(r, t) \omega_\alpha^*(t, f) \frac{dt}{t} \right)^q \, dr \\
\leq \int_0^1 \left( \int_{1-r}^r \omega_\alpha^*(t, f) \frac{dt}{t^\beta} \right)^q (1 - r) \, dr \\
\leq \int_0^1 (\omega_\alpha^*(t, f))^q \frac{1}{t^{1+\alpha q}} \, dt
\]
if \( \alpha < 2 \). This shows that \( f \in \Lambda_\alpha^{p, q} \).

Next, let us assume that \( f \in \Lambda_\alpha^{p, q} \). By Lemma 3.4, for \( 0 < t < 1/2 \),
\[
\omega_\alpha^*(t, f) \lesssim t^2 M_p(1 - t, R^2 f) + \int_0^t \tau M_p(1 - \tau, R^2 f) \, d\tau.
\]
Using Cauchy’s estimate for the first term, changing variables for the second, and applying Hardy’s inequality for the last, we get, for \( \alpha > 0 \),
\[
\int_0^1 (\omega_\alpha^*(t, f))^q \frac{1}{t^{1+\alpha q}} \, dt \\
\leq \sup_{|z| < 1/2} |f(z)|^q + \int_0^1 \frac{t^2 M_p^q(1 - t, R^2 f)}{t^{1+\alpha q}} \, dt \\
+ \int_0^1 \left( \int_0^t \tau M_p(1 - \tau, R^2 f) \, d\tau \right)^q \frac{dt}{t^{1+\alpha q}} \\
\lesssim \|f\|_{H^p}^q + \int_0^1 M_p^q(1 - \tau, R^2 f) \tau^{(2 - \alpha)q - 1} \, d\tau.
\]
This shows that the condition in (3.2) holds. \( \square \)

Roughly speaking, the roles of the indices \( \alpha, p, \) and \( q \) in the mean Lipschitz space \( \Lambda_\alpha^{p, q} \) are as follows. The index \( p \) indicates the basic \( H^p \) norm that is used, the index \( \alpha \) gives the order of smoothness involved, and the index \( q \) represents a rather subtle correction to the order of smoothness. The following result makes these remarks somewhat more precise.

**Proposition 3.6.** Suppose \( \alpha > 0 \) and \( 1 \leq p < \infty \). Then

(i) \( \Lambda_\alpha^{p_1, q_1} \subset \Lambda_\alpha^{p_2, q_2} \) whenever \( 1 \leq q_1 < q_2 \leq \infty \).

(ii) \( \Lambda_\alpha^{p_2, q_2} \subset \Lambda_\alpha^{p_1, q_1} \) if \( 0 < \alpha_1 < \alpha_2 < \infty \). Here \( q_1 \) and \( q_2 \) are arbitrary.

**Proof.** Let \( \sigma_\alpha = \alpha - [\alpha] \) and let \( \rho = (1 - r)/2 \). Then
\[
\left( \int_\rho^1 M_p^{q_1}(r, R^{[\alpha]+1} f)(1 - r)^{(1 - \sigma_\alpha)q_1 - 1} \, dr \right)^{1/q_1} \\
\geq M_p(\rho, R^{[\alpha]+1} f) \left( \int_\rho^1 (1 - r)^{(1 - \sigma_\alpha)q_1 - 1} \, dr \right)^{1/q_1} \\
= M_p(\rho, R^{[\alpha]+1} f)(1 - \rho)^{1 - \sigma_\alpha}.
\]
This proves (i) in the case \( q_2 = \infty \). For \( 1 \leq q_1 < q_2 < \infty \), we have
\[
\int_0^1 M_p^{q_2}(r, R^{[\alpha]+1} f)(1 - r)^{(1 - \sigma_\alpha)q_2 - 1} \, dr \\
\leq \sup_{0 < r < 1} \left( M_p(r, R^{[\alpha]+1} f)(1 - r)^{1 - \sigma_\alpha} \right)^{q_2 - q_1} \\
\times \int_0^1 M_p^{q_1}(r, R^{[\alpha]+1} f)(1 - r)^{(1 - \sigma_\alpha)q_1 - 1} \, dr.
\]
This completes the proof of (i).

By (3.1), we have

$$M_p(r, R^{[\alpha_1]+1}f) \lesssim \sup_{|z|<1/2} |f(z)| + \int_0^r (r-t)^{[\alpha_2]-[\alpha_1]-1} M_p(t, R^{[\alpha_2]+1}f)dt.$$ 

Then the containment in (ii) is proved similarly with the help of the inequality above and the variant of Hardy’s inequality in Lemma 3.7. \(\square\)

Our next result is a characterization of \(\Lambda_{\alpha q}^\ast\) in terms of the fractional differential operators \(R^q\) and \(R^{s,t}\). We need the following variant of Hardy’s inequality.

**Lemma 3.7** ([1]). Let \(\alpha > 0\), \(\beta > 0\), and \(1 \leq p < \infty\). Then we have

$$\int_0^1 (1-r)^{\alpha-1} \left( \int_0^r (r-t)^{\beta-1} F(t) dt \right)^p dr \lesssim \int_0^1 (1-r)^{\alpha+\beta p} F(r)^p dr$$

for all \(F \geq 0\).

**Lemma 3.8** ([1]). Let \(0 < \beta < r\) and let \(\lambda\) be a point in the unit disk \(\mathbb{D}\). Then

$$(1-\lambda)^{\beta r} = \frac{\Gamma(r)}{\Gamma(\beta)\Gamma(r-\beta)} \int_0^1 \frac{t^{r-\beta-1}(1-t)^{\beta-1}}{(1-t\lambda)^r} dt.$$ 

**Lemma 3.9.** Let \(f \in H(\mathbb{D})\), \(\varepsilon > 0\), and \(1-\varepsilon < r < 1\). Then

$$M_p(r, f) \leq \sup_{|z| \leq 1-\varepsilon} |f(z)| + \left(\frac{r}{1-\varepsilon} - 1\right) M_p(r, R^\lambda f).$$

**Proof.** By the fundamental theorem of calculus,

$$|f(z)| \leq \sup_{|z| \leq 1-\varepsilon} |f(z)| + \int_{1-\varepsilon}^1 \frac{1}{t} |R^\lambda f(t\zeta)| dt.$$ 

Therefore,

$$M_p(r, f) \leq \sup_{|z| \leq 1-\varepsilon} |f(z)| + \left(\int_{1-\varepsilon}^r \left(\int_{1-\varepsilon}^1 \frac{1}{t} |R^\lambda f(t\zeta)| dt \right)^p d\sigma(\zeta) \right)^{1/p}.$$ 

By Minkowski’s inequality,

$$M_p(r, f) \leq \sup_{|z| \leq 1-\varepsilon} |f(z)| + \frac{1}{1-\varepsilon} \int_{1-\varepsilon}^r M_p(t, R^\lambda f) dt$$

$$\leq \sup_{|z| \leq 1-\varepsilon} |f(z)| + \left(\frac{r}{1-\varepsilon} - 1\right) M_p(r, R^\lambda f).$$

This proves the desired inequality. \(\square\)

We now characterize the holomorphic mean Lipschitz spaces in terms of the two-parameter fractional differential operators \(R^{s,t}\).

**Theorem 3.10.** Let \(0 < \alpha < \infty\), \(1 \leq p < \infty\) and \(1 \leq q \leq \infty\). Suppose \(t > \alpha\) and \(f \in H^p\). If \(s\) is a real parameter such that none of \(n+s\) and \(n+s+t\) is a negative integer, then \(f \in \Lambda_{\alpha q}^\ast\) if and only if

$$\int_0^1 M_p^q(r, R^{s,t}f)(1-r)^{(t-\alpha)q-1} dr < \infty.$$  (3.3)
Proof. First assume that $f \in \Lambda_{p,q}$. Choose an integer $k$ such that $k > t$ and $s + k > -1$. We have

$$R^{s,t} = R^{s+k,t-k} R^{s,k}.$$ 

It follows that

$$R^{s,t} f(z) = R^{s+k,t-k} \left( R^{s,k} f(z) \right)$$

$$= \lim_{\rho \to 1^{-}} \int_{\mathbb{R}^n} R^{s,k} f(\rho w) \, dv_{s+k}(w)$$

$$= C \lim_{\rho \to 1^{-}} \int_{\mathbb{R}^n} \tau^{s+k+t} (1 - \tau)^{k-t-1} d\tau$$

$$\int_{\mathbb{R}^n} R^{s,k} f(\rho \tau z) \, dv_{s+k}(w)$$

where

$$C = \frac{\Gamma(n+1+s+k)}{\Gamma(k-t) \Gamma(n+1+s+t)}.$$ 

By Lemma 2.6, we have

$$M_p(r, R^{s,t} f) = \left( \int_{\mathbb{R}^n} |R^{s,t} f(r \zeta)|^p \, d\sigma(\zeta) \right)^{1/p}$$

$$\lesssim \lim_{\rho \to 1^{-}} \sum_{j=0}^{k} \left( \int_{\mathbb{R}^n} \left( \int_{0}^{1} (1 - \tau)^{k-t-1} |R^{s,t} f(\rho \tau r \zeta)| \, d\tau \right)^p \, d\sigma(\zeta) \right)^{1/p}$$

$$\lesssim \|f\|_{H^p} + \int_{0}^{1} (1 - \tau)^{k-t-1} M_p(\tau r, R^{s,t} f) \, d\tau.$$ 

By Lemma 3.7, we obtain

$$\int_{0}^{1} M_p^q(r, R^{s,t} f)(1 - r)^{(t-\alpha)q-1} \, dr$$

$$\lesssim \|f\|_{H^p}^q + \int_{0}^{1} \left( \int_{0}^{1} (1 - \tau)^{k-t-1} M_p(\tau r, R^{s,t} f) \, d\tau \right)^q (1 - r)^{(t-\alpha)q-1} \, dr$$

$$\lesssim \|f\|_{H^p}^q + \int_{0}^{1} \left( \int_{0}^{1} (r - \rho)^{k-t-1} M_p(\rho, R^{s,t} f) \, d\rho \right)^q (1 - r)^{(t-\alpha)q-1} \, dr$$

$$\lesssim \|f\|_{H^p}^q + \int_{0}^{1} (1 - r)^{(k-\alpha)q-1} M_p^q(r, R^{s,t} f) \, dr.$$ 

This shows that $f \in \Lambda_{p,q}^{s,t}$ implies condition (3.3).

Next we assume that condition (3.3) holds. Choose an integer $k$ such that $k > t$ and $s + k + t > -1$. We have

$$R^{s,k} = R^{s+k,t} R^{s,k-t}$$

$$R^{s,k+t} = R^{s+k,t} R^{s,t}.$$
Thus
\[
R^{s,k} f(z) = R^{s+k+t,-t}(R^{s,k+t} f)(z)
\]
\[
= \lim_{\rho \to 1^-} \int_{B^n} \frac{R^{s,t+k} f(pw) dv_{s+k+t}(w)}{(1 - \langle z, w \rangle)^{n+1+s+k+t}}
\]
\[
= C \lim_{\rho \to 1^-} \int_0^1 \tau^{n+s+k}(1 - \tau)^{t-1} \int_{B^n} \frac{R^{s,k+t} f(pw) dv_{s+k+t}(w)}{(1 - \langle \tau z, w \rangle)^{n+1+s+k+t}}
\]
\[
= C \lim_{\rho \to 1^-} \int_0^1 \tau^{n+s+k}(1 - \tau)^{t-1} R^{s,t+k} f(\rho \tau z) d\tau,
\]
where
\[
C = \frac{\Gamma(n + 1 + s + k + t)}{\Gamma(t) \Gamma(n + 1 + s + k)}.
\]

Therefore,
\[
M_p(r, R^{s,k} f) = \left( \int_{\mathbb{S}^n} |R^{s,k} f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}
\]
\[
\lesssim \lim_{\rho \to 1^-} \left( \int_{\mathbb{S}^n} \left| \int_0^1 \tau^{n+s+k}(1 - \tau)^{t-1} R^{s,k+t} f(\rho \tau r \zeta) d\tau \right|^p d\sigma(\zeta) \right)^{1/p}
\]
\[
\lesssim \lim_{\rho \to 1^-} \int_0^1 \tau^{n+s+k}(1 - \tau)^{t-1} M_p(\rho \tau r, R^{s,k+t}) d\tau
\]
\[
\lesssim \int_0^1 \tau^{n+s+k}(1 - \tau)^{t-1} M_p(\tau r, R^{s,k+t}) d\tau.
\]

Since \( R^{s,k+t} = R^{s+t,k} R^{s,t} \), it follows from Lemma 2.2 in [6] that for \( \delta = 1 - \tau r \) we have
\[
M_p^p(\tau r, R^{s,k+t} R^{s,t} f) \lesssim M_p^p(\tau r, R^k R^{s,t} f)
\]
\[
\lesssim \frac{1}{(1 - \tau r)^{pk+1}} \int_{\tau r - \delta/4}^{\tau r + \delta/4} M_p^p(x, R^{s,t} f) dx
\]
\[
\lesssim \frac{M_p^p(\tau r + \delta/4, R^{s,t} f)}{(1 - \tau r)^{pk}}.
\]

By Lemma 3.7,
\[
\int_0^1 \frac{1}{M_p^p(r, R^{s,k} f)(1 - r)^{(k-\alpha)q-1}} dr
\]
\[
\lesssim \int_0^1 \left( \int_0^1 (1 - \tau)^{t-1} M_p(\tau r + \delta/4, R^{s,t} f) (1 - \tau r)^k d\tau \right)^q (1 - r)^{(k-\alpha)q-1} dr
\]
\[
\lesssim \int_0^1 \left( \int_0^r (r - \rho)^{t-1} M_p(\rho + (1 - \rho)/4, R^{s,t} f) (1 - \rho)^k d\rho \right)^q (1 - r)^{(k-\alpha)q-1} dr
\]
\[
\lesssim \int_0^1 (1 - r)^{(t-\alpha)q-1} M_p^q(1 - r, R^{s,t} f) dr
\]
\[
\lesssim \int_0^1 (1 - x)^{(t-\alpha)q-1} M_p^q(x, R^{s,t} f) dx.
\]
By Lemmas 2.6 and 3.9,
\[
M_p(r, R^k f) \lesssim \sum_{j=1}^{k-1} M_p(r, R^j f) + M_p(r, R^k f) \\
\lesssim \|f\|_{H^p} + \left( \frac{r}{1 - \varepsilon} - 1 \right) M_p(r, R^k f) + M_p(r, R^k f).
\]
Thus there exists a constant \(c > 0\) that is sufficiently close to 1 such that
\[
M_p(r, R^k f) \lesssim \|f\|_{H^p} + M_p(r, R^k f), \quad c < r < 1.
\]
We conclude that
\[
\int_c^1 M_p^q(r, R^k f)(1 - r)^{(k-\alpha)q-1} \, dr \\
\lesssim \|f\|_{H^p} + \int_c^1 M_p^q(r, R^k f)(1 - r)^{(k-\alpha)q-1} \, dr \\
\lesssim \|f\|_{H^p} + \int_{1/4}^1 M_p^q(r, R^k f)(1 - r)^{(k-\alpha)q-1} \, dr.
\]
This shows that \(f \in \Lambda^{p,q}_{\alpha}\) and completes the proof of the theorem. \(\square\)

A similar characterization of \(\Lambda^{p,q}_{\alpha}\) holds in terms of the one-parameter radial differential operators \(R^t\).

**Theorem 3.11.** Let \(\alpha > 0, 1 \leq p < \infty, \text{ and } 1 \leq q \leq \infty.\) Suppose \(t > \alpha\) and \(f \in H(B^n)\). Then \(f \in \Lambda^{p,q}_{\alpha}\) if and only if
\[
(1 - r)^{t-\alpha} M_p(r, R^k f) \in L^q \left([0, 1), \frac{dr}{1 - r} \right).
\]

**Proof.** If \(t\) is an integer, the desired result is proved in [6]. Thus we assume that \(t\) is not an integer.

First assume that \(f \in \Lambda^{p,q}_{\alpha}\). We choose \(k\) such that \(t < k < t + 1\). Then
\[
R^t f(z) = R^{-(t-\alpha)} R^k f(z) = \frac{1}{\Gamma(k - t)} \int_0^1 \left( \log \frac{1}{\tau} \right)^{k-1-t} R^k f(\tau z) \frac{d\tau}{\tau}.
\]
Since
\[
|R^k f(w)| \lesssim |w| \sum_{|\alpha| \leq k} |\partial^\alpha f(w)|,
\]
it follows from Cauchy’s integral formula that
\[
\int_0^{1/2} \left( \log \frac{1}{\tau} \right)^{k-1-t} |R^k f(\tau z)| \frac{d\tau}{\tau} \lesssim \sup_{|\alpha| \leq k, |z| \leq 1/2} |\partial^\alpha f(z)| \int_0^{1/2} \left( \log \frac{1}{\tau} \right)^{k-1-t} \, d\tau
\]
\[
\lesssim \sup_{|z| \leq 2/3} |f(z)| \int_0^{1/2} (1 - \tau)^{k-1-t} \, d\tau
\]
\[
\lesssim \sup_{|z| \leq 2/3} |f(z)|.
\]
Therefore,
\[
M_p(r, R^t f) \lesssim \sup_{|z| \leq 2/3} |f(z)| + \int_0^1 (1 - \tau)^{k-1-t} M_p(\tau r, R^k f) \, d\tau.
\]
The remaining part for the proof of condition (3.4) is the same as in the proof Theorem 3.10.

Next we assume that condition (3.4) holds. Let \( \sigma_t = t - m \), where \( m = \lfloor t \rfloor \) is the integer part of \( t \). We choose an integer \( k \) such that \( k > t \). Then we have

\[
\mathcal{R}^k f(z) = \mathcal{R}^{-\sigma_t} \mathcal{R}^{k+\sigma_t} f(z) = \frac{1}{t(\sigma_t)} \int_0^1 \left( \log \frac{1}{\tau} \right)^{\sigma_t-1} \mathcal{R}^{k+\sigma_t} f(\tau z) d\tau.
\]

Therefore,

\[
M_p(r, \mathcal{R}^k f) \lesssim \sup_{|z| \leq 2/3} |f(z)| + \int_0^1 (1 - \tau)^{\sigma_t-1} M_p(\tau r, \mathcal{R}^{k+\sigma_t} f) d\tau.
\]

Since \( \mathcal{R}^{k+\sigma_t} = \mathcal{R}^{k-m} \mathcal{R}^t \), it follows from Lemma 2.2 in [6] that for \( \delta = 1 - \tau r \) we have

\[
M_p(\tau r, \mathcal{R}^{k+\sigma_t} f) \lesssim M_p(\tau r + \delta/4, \mathcal{R}^t f) \frac{(1 - \tau r)^{k-m}}{1 - \tau r}.
\]

By the same calculation as in the proof of Theorem 3.10,

\[
\int_0^1 M_p^q(r, \mathcal{R}^k f)(1 - r)^{(k-\alpha)q-1} dr \lesssim \int_0^1 (1 - x)^{(t-\alpha)q-1} M_p^q(x, \mathcal{R}^t f) dx.
\]

This completes the proof of the theorem. □

As a consequence of the theorem above, we show that whenever \( p \) and \( q \) are fixed, the spaces \( \Lambda^{p,q}_{\alpha+\beta} \) are all isomorphic.

**Theorem 3.12.** Let \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \). Suppose \( \alpha > 0 \) and \( \beta > 0 \). Then the linear operator

\[
\mathcal{R}^\beta : \Lambda^{p,q}_{\alpha+\beta} \to \Lambda^{p,q}_\alpha
\]

is bounded, one-to-one, and onto.

**Proof.** Let \( t \) be sufficiently large. For \( f \in H(\mathbb{B}^n) \) we have \( \mathcal{R}^\beta f \in \Lambda^{p,q}_\alpha \) if and only if

\[
(1-r)^{t-\alpha} M_p(r, \mathcal{R}^t(f)) \in L^q \left( [0,1), \frac{dr}{1-r} \right)
\]

if and only if

\[
(1-r)^{t+\beta-(\alpha+\beta)} M_p(r, \mathcal{R}^{t+\beta} f) \in L^q \left( [0,1), \frac{dr}{1-r} \right)
\]

if and only if \( f \in \Lambda^{p,q}_{\alpha+\beta} \). This proves the desired result. □

A similar isomorphism between \( \Lambda^{p,q}_{\alpha+\beta} \) and \( \Lambda^{p,q}_\alpha \) can be constructed using the two-parameter fractional radial differential operators \( \mathcal{R}^{s,t} \).

**Theorem 3.13.** Let \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \). Suppose \( \alpha > 0, \beta > 0 \), and \( f \in H(\mathbb{B}^n) \).

If \( s \) is any real parameter such that none of \( n + s \) and \( n + s + \beta \) is a negative integer, then the operator

\[
\mathcal{R}^{s,\beta} : \Lambda^{p,q}_{\alpha+\beta} \to \Lambda^{p,q}_\alpha
\]

is bounded, one-to-one, and onto.
Proof. Note that the assumptions on $s$ and $\beta$ ensure that the operator $R^{s,\beta}$ is well defined. Let $t$ be a positive number that is large enough so that the operators $R^{s+\beta,t}$ and $R^{s,t+\beta}$ are well defined. By Lemma 2.4, we have

$$R^{s+\beta,t}R^{s,\beta} = R^{s,t+\beta}.$$ 

We may also assume that $t > \alpha$. Then we deduce from Theorem 3.10 that $R^{s,\beta}f \in \Lambda_{\alpha}^{p,q}$ if and only if

$$(1-r)^{t-\alpha}M_p(r, R^{s+\beta,t}(R^{s,\beta}f)) \in L^q \left( [0,1), \frac{dr}{1-r} \right)$$

if and only if

$$(1-r)^{t+\beta-(\alpha+\beta)}M_p(r, R^{s,t+\beta}f) \in L^q \left( [0,1), \frac{dr}{1-r} \right)$$

if and only if $f \in \Lambda_{\alpha+\beta}^{p,q}$. This proves the desired result. \qed

Recall that the special case $p = q = \infty$ gives rise to the classical holomorphic Lipschitz spaces $\Lambda_{\alpha}$ on the unit ball. It is also interesting to consider the special case $p = q < \infty$. In this case, a holomorphic function $f$ in $B^n$ belongs to $\Lambda_{\alpha}^{p,p}$ if and only if

$$\int_{B^n} \left| (1-|z|^2)^{k-\alpha}R^k f(z) \right|^p \frac{dv(z)}{1-|z|^2} < \infty.$$ 

Rewrite the condition as

$$\int_{B^n} \left| (1-|z|^2)^{k}R^k f(z) \right|^p \frac{dv(z)}{(1-|z|^2)^{1+p\alpha}} < \infty.$$ 

If we further specialize to $p\alpha = n$, or $\alpha = n/p$, then $\Lambda_{n/p}^{p,p}$ consists of holomorphic functions $f$ in $B^n$ such that

$$(1-|z|^2)^{k}R^k f(z) \in L^p(B^n, d\lambda),$$

where $k$ is sufficiently large and

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

is the Möbius invariant volume measure on $B^n$. It is well known that the condition above describes the diagonal holomorphic Besov spaces $B_p$ on the unit ball; see [20]. In particular, if $p > n$, the diagonal Besov space $B_p$ consists of holomorphic functions $f$ in $B^n$ such that

$$\int_0^1 \omega^p_p(t,f) \frac{dt}{t^{n+1}} < \infty.$$ 

So the diagonal Besov spaces are actually mean Lipschitz spaces.

4. Hardy Sobolev spaces

Recall that for $p > 0$ and $\alpha > 0$ the Hardy Sobolev space $H^p_\alpha$ consists of $f \in H(B^n)$ such that $R^\alpha f \in H^p$, where $H^p$ is the classical Hardy space. It is well known that $H^p_\alpha \subset H^p$ for all $p > 0$ and $\alpha > 0$ [9]. Therefore, the fractional integral operator $R_\alpha = R^{-\alpha}$ is bounded on $H^p$.

We are going to show that the operator $R^\alpha$ used above can be replaced by any fractional differential operator $R^{s,\alpha}$ of order $\alpha$. We begin with the case when $\alpha$ is a positive integer.
Lemma 4.1. Suppose \( p > 0, \ N \) is a positive integer, and \( s \) is a real parameter such that \( n + s \) is not a negative integer. Then a holomorphic function \( f \) in \( \mathbb{B}^n \) belongs to \( H^p_N \) if and only if \( R^{s,N} f \in H^p \).

Proof. Recall from Lemma 2.5 that there are constants \( c_k, 0 \leq k \leq N, \) such that
\[
R^{s,N} = \sum_{k=0}^{N} c_k R^k,
\]
(4.1)
where \( c_0 = 1 \) and \( c_N \neq 0. \)

First assume that \( f \in H^p_N \), namely, \( R^N f \in H^p. \) It is elementary to check that \( R f \in H^p \) always implies \( f \in H^p. \) Repeating this several times, we see that \( R^k f \in H^p \) for each \( 0 \leq k < N. \) This together with (4.1) shows that \( R^N f \in H^p \) implies that \( R^{s,N} f \in H^p. \)

Next assume that \( R^{s,N} f \in H^p. \) Then using the integral representation for the operator \( R^{s,N} \) and integrating several times shows that \( R^{s+k} f \in H^p \) for each \( 0 \leq k < N. \) Using (4.1) to go up from \( k = 0 \) to \( k = N, \) we then see that each \( R^k f \in H^p \) for each \( 0 \leq k \leq N. \) In particular, \( R^N f \in H^p, \) or \( f \in H^p_N. \) This completes the proof of the lemma.

When \( \alpha \) is not an integer, the proof is more technical.

Theorem 4.2. Suppose \( \alpha > 0, \ p > 0, \) and \( s \) is a real parameter such that none of \( n + s \) and \( n + s + \alpha \) is a negative integer. Then for any \( f \in H(\mathbb{B}^n) \) we have \( f \in H^p_N \) if and only if \( R^{s,\alpha} f \in H^p. \)

Proof. Let \( N \) be a positive integer that is large enough so that for every function \( f \in H^p \) with homogeneous expansion \( f = \sum f_k \) the function \( \sum_{k\geq1} |f_k|/k^{N-\alpha} \) is bounded in \( \mathbb{B}^n. \) According to the asymptotic expansion for a ratio of two gamma functions obtained in [17], there exist a sequence \( \{\gamma_m\} \) of nonzero numbers (with \( \gamma_0 = 1 \)) such that the limit
\[
\lim_{k \to \infty} k^N \left[ k^{-\alpha} \frac{\Gamma(k + 1 + n + s + \alpha)}{\Gamma(k + 1 + n + s)} - \sum_{m=0}^{\infty} \gamma_m k^{-m} \right]
\]
exists. It follows that the limit
\[
\lim_{k \to \infty} k^N \left[ k^{-\alpha} \frac{\Gamma(k + 1 + n + s + \alpha)}{\Gamma(k + 1 + n + s)} - \sum_{m=0}^{N-1} \gamma_m k^{-m} \right]
\]
also exists. Thus there exists \( k_0 \) so that for \( k \geq k_0, \)
\[
\frac{\Gamma(k + 1 + n + s + \alpha)}{\Gamma(k + 1 + n + s)} = \sum_{m=0}^{N-1} \gamma_m k^{\alpha - m} + c_k,
\]
(4.2)
where \( |c_k| \leq C k^{\alpha-N} \) for some constant \( C. \)

Let \( c = \Gamma(n + 1 + s + \alpha)/\Gamma(n + 1 + s) \) and \( T \) be the operator defined by
\[
T f_k = \begin{cases} \left( cR^{s,\alpha} - \sum_{m=0}^{N-1} \gamma_m R^{\alpha - m} \right) f_k, & \text{if } k < k_0, \\ c_k f_k, & \text{if } k \geq k_0, \end{cases}
\]
where \( f_k \) is a homogeneous polynomial of degree \( k. \) Then, by (4.2),
\[
cR^{s,\alpha} f = \sum_{m=0}^{N-1} \gamma_m R^{\alpha - m} f + Tf.
\]
(4.3)
Since the assumption on \( N \) ensures that \( T \) is bounded on \( H^p \), and since \( H^p_\alpha \subset H^p \) for all \( p > 0 \) and \( \alpha > 0 \), it suffices for us to show that, whenever \( f \in H^p \), we have \( R^{\alpha \cdot} f \in H^p \) if and only if \( R^\alpha f \in H^p \).

First assume that \( R^\alpha f \in H^p \), then \( R^\beta f \in H^p \) for any \( \beta < \alpha \). In fact, we can write the condition \( R^\alpha f \in H^p \) as \( R^{\alpha - \beta} R^\beta f \in H^p \), which implies that \( R^\beta f \in H^p_{\alpha - \beta} \subset H^p \).

In particular, we have \( R^{\alpha - m} f \in H^p \) for all \( 0 \leq m < N \). Since \( T f \) is in \( H^p \) as well, we conclude that \( e R^\alpha f \in H^p \), or \( R^\alpha f \in H^p \).

Next we assume that \( R^{\alpha \cdot} f \in H^p \). Then by (4.3), the function

\[
\gamma_0 R^{\alpha} f + \gamma_1 R^{\alpha - 1} f + \cdots + \gamma_N R^{\alpha - N} f \in H^p.
\]

Since the fractional integral operator \( R^{-1} \) is bounded on \( H^p \), we see that the function

\[
\gamma_0 R^{\alpha - 1} f + \cdots + \gamma_N R^{\alpha - N} f \in H^p.
\]

But by the choice of \( N \), the function \( R^{\alpha - N} f \in H^p \). So the function

\[
\gamma_0 R^{\alpha - 1} f + \cdots + \gamma_N R^{\alpha - N + 1} f \in H^p.
\]

Repeating this process, we will eventually obtain \( R^{\alpha - N + 1} f \in H^p \).

Iterate the arguments in the previous paragraph. Eventually we will obtain \( R^\alpha f \in H^p \). This completes the proof of the theorem. \( \square \)

As a consequence of the above theorem we obtain the following integral representation for Hardy Sobolev functions, which is an extension of Theorem 2.1 in [5] to the full range \( \alpha > 0 \), except the obviously singular cases.

**Corollary 4.3.** Suppose \( p > 1 \), \( \alpha > 0 \), and \( \alpha \neq n, n + 1, n + 2, \ldots \). Then for any \( f \in H(B^n) \) we have \( f \in H^p_\alpha \) if and only if there exists a function \( g \in L^p(S^n, d\sigma) \) such that

\[
f(z) = \int_{S^n} \frac{g(\zeta) \, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^{n-\alpha}}
\]

for all \( z \in B^n \).

**Proof.** If \( f \) is represented as the integral in (4.4), we apply the operator \( R^{-(\alpha + 1)} \cdot \) to both sides in (4.4) and use Lemma 2.1 to obtain

\[
R^{-(\alpha + 1)} \cdot f = \int_{S^n} \frac{g(\zeta) \, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^n}, \quad z \in B^n.
\]

Since \( g \in L^p(S^n, d\sigma) \) and \( p > 1 \), we conclude that \( R^{-(\alpha + 1)} \cdot f \in H^p \). By Theorem 4.2, we get \( f \in H^p_\alpha \).

Conversely, if \( f \in H^p_\alpha \), then it follows from Theorem 4.2 that the function \( g = R^{-(\alpha + 1)} \cdot f \in H^p \subset L^p(S^n, d\sigma) \). By the reproducing property of the Cauchy kernel, we have

\[
R^{-(\alpha + 1)} \cdot f(z) = \int_{S^n} \frac{g(\zeta) \, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^n}, \quad z \in B^n.
\]

Apply the operator \( R_{-(\alpha + 1)} \cdot \) to both sides above and use Lemma 2.1. We obtain

\[
f(z) = \int_{S^n} \frac{g(\zeta) \, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^{n-\alpha}}
\]

for all \( z \in B^n \). \( \square \)
Next we are going to obtain characterizations of $H^p_\alpha$ in terms of various maximal functions. More specifically, we consider the radial maximal function

$$(M_{\text{rad}} f)(\zeta) = \sup\{|f(r\zeta)| : 0 \leq r < 1\}, \quad \zeta \in S,$$

and the admissible maximal function

$$(M_{\theta} f)(\zeta) = \sup\{|f(z)| : z \in D_\theta(\zeta)\},$$

where $\theta > 1$ and

$$D_\theta(\zeta) = \left\{ z \in B : |1 - \langle z, \zeta \rangle| < \frac{\theta}{2}(1 - |z|^2) \right\},$$

denotes an admissible approach region. We will also use the admissible area function

$$A_{\theta} f(\zeta) = \left( \int_{D_\theta(\zeta)} |R f(z)|^2 (1 - |z|)^{1-n} \, dv(z) \right)^{1/2},$$

and the Littlewood-Paley $g$-function

$$g(f)(\zeta) = \left( \int_0^1 |R f(r\zeta)|^2 (1 - r) \, dr \right)^{1/2}.$$

More generally, we define an admissible area function

$$(A^{s,t}_{\theta,\gamma} f)(\zeta) = \left( \int_{D_\theta(\zeta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2\gamma-1-n} \, dv(z) \right)^{1/2},$$

and we define the Littlewood-Paley $g$-function of order $t$ by

$$g^{s,t}_t(f)(\zeta) = \left( \int_0^1 |R^{s,t} f(r\zeta)|^2 (1 - r)^{2\gamma-1} \, dr \right)^{1/2}.$$

**Proposition 4.4.** Let $0 < \alpha < t$. The following conditions are equivalent.

(a) $f \in H^p_\alpha$.
(b) $A^{t-\alpha}_{\theta,t} f(\zeta) \in L^p(d\sigma)$.
(c) $g^{t-\alpha}_t(f)(\zeta) \in L^p(d\sigma)$.

**Proof.** This is proved in [1]. □

For the two-parameter fractional differential operators $R^{s,t}$ we define the corresponding admissible area function by

$$A^{s,t}_{\theta,\gamma} f(\zeta) = \left( \int_{D_\theta(\zeta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2\gamma-1-n} \, dv(z) \right)^{1/2},$$

and the corresponding Littlewood-Paley $g$-function by

$$g^{s,t}_t(f)(\zeta) = \left( \int_0^1 |R^{s,t} f(r\zeta)|^2 (1 - r)^{2\gamma-1} \, dr \right)^{1/2}.$$

For $0 < r < 1$ and $\eta \in S^n$ we let

$$S_\theta(r, \eta) = \left\{ z \in D_\theta(\eta) : \frac{1}{2}(1 - r^2) < 1 - |z|^2 < 2(1 - r^2) \right\}.$$

We will show that Proposition 4.4 above can be generalized to the two-parameter case.
Lemma 4.5. Let \( t > 0, \gamma > 0, 1 < \theta_1 < \theta_2, \) and \( \eta \in \mathbb{S}^n. \) Let \( s \) be a real parameter such that none of \( n + s \) and \( n + s + t \) is a negative integer. Then
\[
\int_{D_{\theta_1}(\eta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2\gamma-n-1} \, dv(z) \lesssim \int_{D_{\theta_2}(\eta)} |f(z)|^2 (1 - |z|)^{2(\gamma-t)-n-1} \, dv(z).
\]

Proof. Choose an integer \( k \) such that \( k > t \) and \( s + k > -1. \) We have
\[
R^{s,t} = R^{s+k,t-k} R^{s,k}.
\]
By the same calculation as in the proof of Theorem 3.10, for \( r \zeta \in D_{\theta_1}(\eta) \) it follows that
\[
|R^{s,t} f(r\zeta)| \lesssim \lim_{\rho \to 1^-} \int_0^1 (1 - \tau)^{(k-t)-1} |R^{s,k} f(\rho \tau r \zeta)| \, d\tau.
\]
By Lemma 2.5 and Lemma 3.5 in [1], we have
\[
|R^{s,k} f(\rho \tau r \zeta)| \lesssim \sum_{j=0}^k |R^j f(\rho \tau r \zeta)|
\]
\[
\lesssim \left( \frac{1}{(1 - \rho \tau r \zeta)^{n+1+2k}} \int_{S_{\rho \tau r \zeta}(\rho \tau r, \eta)} |f(w)|^2 \, dv(w) \right)^{1/2}.
\]
Thus we have
\[
|R^{s,t} f(r\zeta)| \lesssim \int_0^1 (1 - \tau)^{(k-t)-1} \left( \frac{1}{(1 - \tau r)^{n+1+2k}} \int_{S_{\tau r}(\tau r, \eta)} |f(w)|^2 \, dv(w) \right)^{1/2} \, d\tau.
\]
Therefore
\[
\int_{D_{\theta_1}(\eta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2\gamma-n-1} \, dv(z)
\]
\[
\lesssim \int_0^1 d\tau \int_{\{\zeta \in \mathbb{S}^n : r \zeta \in D_{\theta_1}(\eta)\}} |R^{s,t} f(r\zeta)|^2 (1 - r)^{2\gamma-n-1} d\sigma(\zeta)
\]
\[
\lesssim \int_0^1 (1 - r)^{2\gamma-n-1} \, dr \cdot \left[ \int_0^r (r - x)^{(k-t)-1} \left( \frac{1}{(1 - x)^{n+1+2k}} \int_{S_{x}(x, \eta)} |f(w)|^2 \, dv(w) \right)^{1/2} \, dx \right]^2
\]
\[
\lesssim \int_0^1 (1 - r)^{(2\gamma-t)-n-1} \, dr \int_{S_{\rho \tau r}(\rho \tau r, \eta)} |f(w)|^2 \, dv(w)
\]
\[
\lesssim \int_{D_{\theta_2}(\eta)} (1 - |z|)^{(2\gamma-t)-n-1} |f(z)|^2 \, dv(z).
\]
This proves the lemma.

Lemma 4.6. Let \( t > 0, \gamma > 0, 1 < \theta_1 < \theta_2, \) and \( \eta \in \mathbb{S}^n. \) Let \( s \) be a real parameter such that none of \( n + s \) and \( n + s + t \) is a negative integer. Then
\[
\int_{D_{\theta_1}(\eta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2\gamma-n-1} \, dv(z) \lesssim \int_{D_{\theta_2}(\eta)} |f(z)|^2 (1 - |z|)^{2(\gamma+t)-n-1} \, dv(z).
\]
Proof. Choose a positive integer \( k \) such that \( s + k > -1 \). We have
\[
R_{s,t} = R^{s+t,-t} = R^{s+k, -k-t} R^{s,k}.
\]
By the same calculation as in the proof of Lemma 4.5, we have the result. \( \square \)

Corollary 4.7. Let \( t_1, t_2 \in \mathbb{R} \) and \( \gamma_1, \gamma_2 > 0 \) be such that \( t_2 - t_1 = \gamma_2 - \gamma_1 \). Let \( s \) be a real parameter such that none of \( n + s, n + s + t_1, \) and \( n + s + t_2 \) is a negative integer. Then \( A^{\gamma_1}_{\theta}(f) \) and \( A^{\gamma_2}_{\theta}(f) \) are pointwise equivalent, up to replacement of the apertures.

Lemma 4.8 ([11]). Let \( f \in H(\mathbb{B}^n) \) and \( \zeta \in \mathbb{S}^n \). Fix \( 1 < \theta_1 < \theta_2, \mu, \nu \geq 0, \) and \( \gamma > 0 \). Then
\[
\int_{D_{\vartheta_1}(\zeta)} |R^{-\mu} R^\nu f(z)|^2 (1 - |z|)^{2\gamma - n - 1} \, dv(z) \lesssim \int_{D_{\vartheta_2}(\zeta)} |f(z)|^2 (1 - |z|)^{2(\gamma - \nu + \mu) - n - 1} \, dv(z).
\]

Lemma 4.9 ([11]). Let \( f \in H(\mathbb{B}^n), \gamma > 0, \) and \( \zeta \in \mathbb{S}^n \). Then
\[
\int_0^1 |f(r\zeta)|^2 (1 - r)^{2\gamma - 1} \, dr \lesssim \int_{D_{\vartheta}(\zeta)} |f(z)|^2 (1 - |z|)^{2\gamma - 1 - n} \, dv(z).
\]

Lemma 4.10 ([1]). There exists a constant \( c > 0 \) such that
\[
\|M_\theta f\|_{L^p(\sigma)} \leq c \|M_{\text{rad}} f\|_{L^p(\sigma)}
\]
for all \( f \in H^p \).

Lemma 4.11. Let \( t_1, t_2 \in \mathbb{R} \) and \( \gamma_1, \gamma_2 > 0 \) be such that \( t_2 - t_1 = \gamma_2 - \gamma_1 \). Let \( s \) be a real parameter such that none of \( n + s, n + s + t_1, \) and \( n + s + t_2 \) is a negative integer. Then the conditions \( g^{s,t_1}_{\gamma_1} f \in L^p(\sigma) \) and \( g^{s,t_2}_{\gamma_2} f \in L^p(\sigma) \) are equivalent. When \( t_1 = \gamma_1 \), these conditions are also equivalent to \( M_\theta f \in L^p(\sigma) \).

Proof. Consider the following area function in the real variable theory,
\[
S^{s,t}_{\theta,\gamma_1} f(\zeta) = \left( \int_{\Gamma_{\theta}(\zeta)} |R^{s,t} f(z)|^2 (1 - |z|)^{2(\gamma - n)} \, dv(z) \right)^{1/2}, \quad \gamma > 0,
\]
where \( \Gamma_{\theta}(\zeta) \) is the cone of aperture \( \theta \) and vertex \( \zeta \). By the same methods used in the proof of Lemmas 4.5 and 4.6, we can show that the functions \( S^{s,t_1}_{\theta,\gamma_1} f \) and \( S^{s,t_2}_{\theta,\gamma_2} f \) are pointwise equivalent up to replacement of the apertures.

By Lemma 4.9, we have
\[
g^{s,t_1}_{\gamma_1} f(\zeta) \lesssim S^{s,t_1}_{\theta,\gamma_1} f(\zeta).
\]
On the other hand, by the same arguments used in ([7], Cor.3 p.171), but using the Hilbert space \( L^2(0,1) \) with the measure \( (1 - r)^\gamma \, dr \), we can show that if \( g^{s,t_1}_{\gamma_1} f \in L^p(\sigma) \), then
\[
\left( \int_{\Gamma_{\theta}(\zeta)} |R R^{s,t_1} f(z)|^2 (1 - |z|)^{2(\gamma + 1 - n)} \, dv(z) \right)^{1/2} \in L^p(\sigma).
\]
By Lemma 4.8,
\[
\left( \int_{\Gamma_{\theta}(\zeta)} |R R^{s,t_1} f(z)|^2 (1 - |z|)^{2(\gamma + 1 - n)} \, dv(z) \right)^{1/2} \sim S^{s,t_1}_{\theta,\gamma_1} f(\zeta).
\]
Therefore, the conditions \( g^{s,t_1}_{\gamma_1} f \in L^p(\sigma) \) and \( S^{s,t_1}_{\theta,\gamma_1} f \in L^p(\sigma) \) are equivalent. \( \square \)
We can now characterize the spaces $H^p_\alpha$ in terms of the radial maximal function, the admissible maximal function, the admissible area function, and the generalized Littlewood-Paley $g$-function.

**Theorem 4.12.** Let $0 < \alpha < t$ and $f \in H^p$. Let $s$ be a real parameter such that none of $n + s$, $n + s + t$, and $n + s + \alpha$ is a negative integer. The following conditions are equivalent.

(a) $f \in H^p_\alpha$.
(b) $A_{p,1-\alpha}^s f \in L^p(d\sigma)$.
(c) $g_{t-\alpha}(f) \in L^p(d\sigma)$.
(d) $M_{\text{rad}}(R^{s,\alpha} f) \in L^p(d\sigma)$.
(e) $M_{\theta}(R^{s,\alpha} f) \in L^p(d\sigma)$.

**Proof.** To prove the equivalence of (a) and (b), we write

$$R^{s,t} = R^{-(1+\alpha)} R^{s,t} R^{1+\alpha}$$

and apply Lemmas 4.8 and 4.5 to obtain one direction of the equivalence. In the other direction, we write

$$R^{1+\alpha} = R^{1+\alpha} R_{s,t} R^{s,t}$$

and apply Lemmas 4.8 and 4.6.

That condition (b) implies (c) follows from Lemma 4.9.

To show that condition (c) implies (d), take $t = \alpha + 1$. Then

$$g^{s,\alpha+1}_t f(\zeta) = \left( \int_0^1 |R^{s,\alpha+1} f(r\zeta)|^2 (1-r) \, dr \right)^{1/2} \in L^p(d\sigma).$$

Since

$$R^{s,\alpha+1} = R^{s+\alpha,1} R^{s,\alpha} = R^{s,\alpha} + \frac{1}{n+s+t+1} R R^{s,\alpha},$$

we have

$$\int_0^1 |RR^{s,\alpha} f(r\zeta)|^2 (1-r) \, dr \lesssim \int_0^1 |R^{s,\alpha+1} f(r\zeta)|^2 (1-r) \, dr + \int_0^1 |R^{s,\alpha} f(r\zeta)|^2 (1-r) \, dr.$$

Note that

$$|R^{s,\alpha} f(r\zeta)| \lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)| + \int_0^r |RR^{s,\alpha} f(t\zeta)| \, dt.$$

Therefore, by Lemma 3.7,

$$\int_0^1 |R^{s,\alpha} f(r\zeta)|^2 (1-r) \, dr \lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)|^2 + \int_0^1 \left( \int_0^r |RR^{s,\alpha} f(t\zeta)| \, dt \right)^2 (1-r) \, dr \lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)|^2 + \int_0^1 (1-r)^3 |RR^{s,\alpha} f(r\zeta)|^2 \, dr.$$
Thus

\[
\int_0^1 (1-c(1-r)^2)|\mathcal{R}R^{s,\alpha} f(r\zeta)|^2(1-r) \, dr \\
\lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)|^2 + \int_0^1 |R^{s,\alpha+1} f(r\zeta)|^2(1-r) \, dr.
\]

If we choose \( r_0 > 0 \) close enough to 1, then

\[
\int_{r_0}^1 |\mathcal{R}R^{s,\alpha} f(r\zeta)|^2(1-r) \, dr \lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)|^2 + \int_0^1 |R^{s,\alpha+1} f(r\zeta)|^2(1-r) \, dr.
\]

It follows that

\[
\left( \int_0^1 |\mathcal{R}R^{s,\alpha} f(r\zeta)|^2(1-r) \, dr \right)^{1/2} \lesssim \sup_{|z| \leq 1/2} |R^{s,\alpha} f(z)|
\]

\[
+ \left( \int_0^1 |R^{s,\alpha+1} f(r\zeta)|^2(1-r) \, dr \right)^{1/2},
\]

so the function

\[
g(R^{s,\alpha} f)(\zeta) = \left( \int_0^1 |\mathcal{R}R^{s,\alpha} f(r\zeta)|^2(1-r) \, dr \right)^{1/2}
\]

belongs to \( L^p(d\sigma) \). This shows that \( R^{s,\alpha} f \in H^p \) and \( M_{\text{rad}}(R^{s,\alpha} f) \in L^p(d\sigma) \).

That condition (d) implies (e) follows from Lemma 4.10.

Finally, to show that condition (e) implies (a), we recall from [1] that if \( M_0(R^{s,\alpha} f) \in L^p(d\sigma) \), then \( A_{\theta_2}(R^{s,\alpha} f) \in L^p(d\sigma) \) for \( \theta_2 < \theta \). By Lemma 4.8,

\[
A_{\theta_2}^{s,\alpha+1} f(\zeta) = \left( \int_{D_{\theta_2}(\zeta)} |R^{s,\alpha+1} f(z)|^2(1-|z|)^{1-n} \, dv(z) \right)^{1/2}
\]

\[
\lesssim \left( \int_{D_{\theta_2}(\zeta)} |R^{s,\alpha} f(z)|^2(1-|z|)^{1-n} \, dv(z) \right)^{1/2}
\]

\[
+ \left( \int_{D_{\theta_2}(\zeta)} |\mathcal{R}R^{s,\alpha} f(z)|^2(1-|z|)^{1-n} \, dv(z) \right)^{1/2}
\]

\[
\lesssim A_{\theta_2}(R^{s,\alpha} f) \in L^p(d\sigma).
\]

This along with the condition (b) of \( t = \alpha + 1 \) shows that \( f \in H^p_B \). The proof of the theorem is now complete. \( \square \)

5. Comparison of Mean Lipschitz Spaces with Hardy Sobolev Spaces

For \( z \in B \) and \( \delta > 0 \) let \( P(z,\delta) \) be the non-isotropic polydisc defined as follows. If \( z = r\zeta, 0 \leq r < 1, \) and \( \zeta \in \mathbb{S}^n, \) pick \( \eta_2, \ldots, \eta_n \) so that \( \{\zeta, \eta_2, \ldots, \eta_n\} \) is an orthonormal basis of \( \mathbb{C}^n \). Then define

\[
P(z,\delta) = \left\{ w = r\zeta + \lambda \zeta + \sum_{j=2}^n \lambda_j \eta_j : |\lambda| < \delta, |\lambda_j| < \delta^{1/2}, j = 2, \ldots, n \right\}.
\]
Lemma 5.1. Let $1/2 < r < 1$, $\delta = 1 - r$, and

$$Q(\zeta, \delta) = \{ \eta \in S : |1 - \langle \eta, \zeta \rangle| < \delta \}.$$ 

Then there is some $\varepsilon > 0$ such that

$$P(r\zeta, \varepsilon\delta) \subset \left\{ t\eta : r - \frac{\delta}{4} < t < r + \frac{\delta}{4}, \eta \in Q(\zeta, \delta) \right\}.$$

Proof. Let $w \in P(r\zeta, \varepsilon\delta)$, where $\varepsilon > 0$ will be determined later. Then

$$w = (r + \lambda)\zeta + \sum_{j=2}^{n} \lambda_j \eta_j,$$

where $|\lambda| < \varepsilon\delta$ and $|\lambda_j| < (\varepsilon\delta)^{1/2}$ for $j = 2, \ldots, n$. Thus

$$|w|^2 \leq |r + \lambda|^2 + \sum_{j=2}^{n} |\lambda_j|^2 \leq r^2 + |\lambda|^2 + 2|\lambda| + (n - 1)\varepsilon\delta \leq r^2 + (n + 2)\varepsilon\delta.$$

Also

$$|w|^2 \geq (r - |\lambda|)^2 = r^2 - 2r|\lambda| + |\lambda|^2 \geq r^2 - 2r|\lambda| \geq r^2 - 2\varepsilon\delta.$$

So

$$r^2 - 2\varepsilon\delta \leq |w|^2 \leq r^2 + (n + 2)\varepsilon\delta.$$

If we choose $\varepsilon$ such that

$$0 < \varepsilon < \frac{r}{2(n + 2)} + \frac{\delta}{16(n + 2)},$$

then

$$\left(r - \frac{\delta}{4}\right)^2 = r^2 - 2\delta \left(\frac{r}{4} - \frac{\delta}{32}\right) < r^2 - 2\delta \left(\frac{r}{2(n + 2)} + \frac{\delta}{16(n + 2)}\right) r^2 - 2\varepsilon\delta \leq |w|^2,$$

and

$$\left(r + \frac{\delta}{4}\right)^2 = r^2 + (n + 2)\delta \left(\frac{r}{2(n + 2)} + \frac{\delta}{16(n + 2)}\right) > r^2 + (n + 2)\delta\varepsilon \geq |w|^2.$$

Thus we have

$$r - \frac{\delta}{4} < |w| < r + \frac{\delta}{4}.$$ 

On the other hand,

$$|1 - \langle w/|w|, \zeta \rangle| = |1 - (r + \lambda)/|w|| \leq |1 - r/|w|| + |\lambda||/|w|$$

$$< \left\{ \begin{array}{ll}
\frac{\delta}{|w|-\varepsilon} + \frac{4\delta}{|w|-\varepsilon}, & \text{if } r - \frac{\delta}{4} < |w| \leq r, \\
\frac{\delta}{3r+1} + \frac{\varepsilon}{r}, & \text{if } r \leq |w| < r + \frac{\delta}{4}.
\end{array} \right.$$ 

Since

$$\frac{3r^2}{3r + 1} > \frac{5r - 2}{4} > \frac{r}{2(n + 2)} + \frac{\delta}{16(n + 2)} > \varepsilon,$$

we have

$$\frac{1 + 4\varepsilon}{5r - 1} < 1, \quad \frac{1}{3r + 1} + \frac{\varepsilon}{r} < 1.$$

Thus

$$|1 - \langle w/|w|, \zeta \rangle| < \delta \quad \text{for } w \in P(r\zeta, \varepsilon\delta).$$
Therefore,
\[ r - \frac{\delta}{4} < |w| < r + \frac{\delta}{4} \quad \text{and} \quad \frac{w}{|w|} \in Q(\zeta, \delta). \]
The lemma is proved. \(\square\)

Our next result compares the spaces \(H^p_\alpha\) and \(\Lambda^{p,q}_\alpha\) in various cases.

**Theorem 5.2.** Let \(1 < p < \infty\) and \(\alpha > 0\). We have
(a) \(H^p_\alpha \subset \Lambda^{p,p}_\alpha\) if \(p \geq 2\).
(b) \(H^p_\alpha \subset \Lambda^{p,2}_\alpha\) if \(p \leq 2\).
(c) \(\Lambda^{p,p}_\alpha \subset H^p_\alpha\) if \(p \leq 2\).
(d) \(\Lambda^{p,2}_\alpha \subset H^p_\alpha\) if \(p \geq 2\).

**Proof.** Let \(\sigma = \alpha - \lfloor \alpha \rfloor\) and consider the following variants of the Littlewood-Paley \(g\)-function:
\[ g^{s,t,p}_\gamma f(\zeta) = \left( \int_0^1 |R^{s,t} f(r\zeta)|^p (1 - r)^{p\gamma - 1} dr \right)^{1/p} \]
if \(p < \infty\), and
\[ g^{s,t,\infty}_\gamma f(\zeta) = \sup_{0 < r < 1} (1 - r)^\gamma |R^{s,t} f(r\zeta)|. \]
For \(\zeta \in S^n\) and \(t > \alpha\) the non-isotropic Hardy-Littlewood maximal operator \(M_{HL}\) is defined by
\[ M_{HL} f(\zeta) = \sup_{\delta > 0} \frac{1}{\sigma(Q(\zeta, \delta))} \int_{Q(\zeta, \delta)} |f(\eta)| d\sigma(\eta). \]
Let \(1/2 < r < 1\) and \(\delta = 1 - r\). By Lemma 5.1, there is an \(\varepsilon > 0\) such that
\[ P(r\zeta, \varepsilon \delta) \subset \left\{ t\eta : r - \frac{\delta}{4} < t < r + \frac{\delta}{4}, \eta \in Q(\zeta, \delta) \right\}. \]
By the subharmonicity of \(|R^{s,t} f(z)|^2\), we have
\[ |R^{s,t} f(r\zeta)| \lesssim \left( \frac{1}{\delta^{n+1}} \int_{P(r\zeta, \varepsilon \delta)} |R^{s,t} f(z)|^2 dv(z) \right)^{1/2}. \]
Let \(w = t\eta\) with \(t = |w|\) and \(\eta \in S\). Then
\[ dv(w) = c \, d\sigma(\eta) t^{2n-1} dt. \]
Thus
\[ |R^{s,t} f(r\zeta)| (1 - r)^{t-\alpha} \]
\[ \lesssim \left( \frac{1}{\delta^{n+1}} \int_{r - \delta/4}^{r + \delta/4} dt \int_{Q(\zeta, \delta)} |R^{s,t} f(t\eta)|^2 d\sigma(\eta) \right)^{1/2} \]
\[ \lesssim \left( \frac{1}{\delta^n} \int_{r - \delta/4}^{r + \delta/4} dt \int_{Q(\zeta, \delta)} |R^{s,t} f(t\eta)|^2 (1 - t)^{2(t-\alpha)-1} d\sigma(\eta) \right)^{1/2} \]
\[ \lesssim \left( M_{HL} \left( \int_0^1 |R^{s,t} f(t\eta)|^2 (1 - t)^{2(t-\alpha)-1} dt \right)(\zeta) \right)^{1/2}. \]
Therefore,
\[ \int_{S^n} |g_{t-\alpha}^{s,t} f(\zeta)|^p \, d\sigma(\zeta) \leq \int_{S^n} (M_{H\ell}(g_{t-\alpha}^{s,t} f)^2(\zeta))^{p/2} \, d\sigma(\zeta) \]
\[ \lesssim \int_{S^n} (g_{t-\alpha}^{s,t} f(\zeta))^p \, d\sigma(\zeta) \lesssim \|f\|^p_{H^n_\alpha}. \]

It is clear that for \( p \geq 2 \) we have
\[ g_{t-\alpha}^{s,t,p} f(\zeta) \leq (g_{t-\alpha}^{s,t} f(\zeta))^{2/p} (g_{t-\alpha}^{s,t,\infty} f(\zeta))^{1-2/p}. \]

By Hölder’s inequality,
\[ \|g_{t-\alpha}^{s,t,p} f\|_{L^p(\partial\sigma)} \leq \|g_{t-\alpha}^{s,t} f\|_{L^p(\partial\sigma)} \|g_{t-\alpha}^{s,t,\infty} f\|_{L^p(\partial\sigma)} \lesssim \|f\|_{H^n_\alpha}, \]
and by Fubini’s theorem,
\[ \|g_{t-\alpha}^{s,t,p} f\|_{L^p(\partial\sigma)} = \int_{S^n} \int_0^1 |R^{s,t} f(r\zeta)|^p (1-r)^{p(t-\alpha)-1} \, dr \, d\sigma(\zeta) \]
\[ = \int_0^1 \int_{S^n} |R^{s,t} f(r\zeta)|^p \, d\sigma(\zeta) (1-r)^{p(t-\alpha)-1} \, dr \]
\[ = \int_0^1 M_p^p(r, R^{s,t})(1-r)^{p(t-\alpha)-1} \, dr. \]

This proves assertion (a).

To prove (b), we begin with Minkowski’s inequality. Thus for \( p \leq 2 \), we have
\[ \int_0^1 \left( \int_{S^n} |R^{s,t} f(r\zeta)|^p \, d\sigma(\zeta) \right)^{2/p} (1-r)^{2(t-\alpha)-1} \, dr \]
\[ \leq \left( \int_{S^n} \left( \int_0^1 |R^{s,t} f(r\zeta)|^2 (1-r)^{2(t-\alpha)-1} \, dr \right)^{p/2} \, d\sigma(\zeta) \right)^{2/p} \]
\[ \leq \left( \int_{S^n} (g_{t-\alpha}^{s,t} f(\zeta))^p \, d\sigma(\zeta) \right)^{2/p} \lesssim \|f\|_{H^n_\alpha}^2. \]  
\[ (5.1) \]

Thus \( H^n_p \subset \Lambda^{p,2}_\alpha \) when \( p \leq 2 \).

When \( p \geq 2 \), Minkowski’s inequality in (5.1) is reversed, namely,
\[ \|f\|_{H^n_\alpha} \lesssim \left( \int_{S^n} \left( \int_0^1 |R^{s,t} f(r\zeta)|^2 (1-r)^{2(t-\alpha)-1} \, dr \right)^{p/2} \, d\sigma(\zeta) \right)^{2/p} \]
\[ \leq \int_0^1 \left( \int_{S^n} |R^{s,t} f(r\zeta)|^p \, d\sigma(\zeta) \right)^{2/p} (1-r)^{2(t-\alpha)-1} \, dr. \]

This shows that \( \Lambda^{p,2}_\alpha \subset H^n_p \) for \( p \geq 2 \) and proves assertion (d).

Finally, if \( 1 < p \leq 2 \), then
\[ (g_{t-\alpha}^{s,t} f)(\zeta) \leq (g_{t-\alpha}^{s,t,p} f(\zeta))^{p/2} (g_{t-\alpha}^{s,t,\infty} f(\zeta))^{1-p/2}. \]

By Hölder’s inequality,
\[ \|g_{t-\alpha}^{s,t,p} f\|_{L^p(\partial\sigma)} \leq \|g_{t-\alpha}^{s,t} f\|_{L^p(\partial\sigma)} \|g_{t-\alpha}^{s,t,\infty} f\|^1_{L^p(\partial\sigma)}. \]
Therefore, \( \Lambda^{p,p}_\alpha \subset H^n_p \). This proves (c) and completes the proof of the theorem. \( \square \)
Remark 5.3. A special consequence of the theorem above is that \( \Lambda^2_\alpha = H^2_\alpha \). This follows from the definitions directly. Also, by (i) of Proposition 3.6 and (a), (b) of Theorem 5.2, we have \( H^p_\alpha \subset \Lambda^p_\alpha \) for any \( 1 < p < \infty \) and \( \alpha > 0 \). However, the converse inclusion is not true. A counter-example will be given in Theorem 5.5.

Lemma 5.4 ([14]). Let
\[
L_{j,k} = \mathbb{R}^j \times \mathbb{C}^k \times \{0\} \times \cdots \times \{0\} \subset \mathbb{C}^n,
\]
where \( 1 \leq j \leq n \) and \( 1 \leq j + k \leq n \). For \( 0 < p < \infty \) we have
\[
\int_{B^n \cap L_{j,k}} |f(z)|^p (1 - |z|^2)^{n-1/2(j+2k+1)} \, dz \lesssim \|f\|^p_{H^p},
\]
where \( dz \) is Lebesgue measure on \( L_{j,k} \).

Theorem 5.5. Let \( 1 \leq p < \infty \) and \( \alpha > 0 \). If \( \alpha \leq n/p \) and \( \alpha \) is not an integer, then there is a function in \( \Lambda^p_\alpha \) that is not in \( H^p_\alpha \).

Proof. Let \( m = \lfloor \alpha \rfloor \) and \( \sigma = \alpha - m \). If \( \alpha \) is not an integer, then \( 0 < \sigma < 1 \). We define
\[
f(z) = \int_0^1 \frac{1}{t^{1-\sigma}(1 + t - z_1)^{n/p-m}} \, dt, \quad z \in \mathbb{B}^n.
\]
Then \( f \in H(\mathbb{B}^n) \) and \( \partial^j f(z)/\partial z_1^j \) is equal to
\[
\left( \frac{n}{p} - m \right) \left( \frac{n}{p} - m + 1 \right) \cdots \left( \frac{n}{p} - m + j - 1 \right) \int_0^1 \frac{1}{t^{1-\sigma}(1 + t - z_1)^{n/p-m+j}} \, dt.
\]
Since \( f \) is a function of \( z_1 \) only, we have
\[
\mathcal{R}^{m+1} f = \sum_{j=1}^{m+1} c_j z_1^j \frac{\partial^j f}{\partial z_1^j}, \quad c_j > 0.
\]
By Minkowski’s inequality, we have
\[
M_p(r, \mathcal{R}^{m+1} f) \lesssim \sum_{j=1}^{m+1} \left( \int_{S^n} \left| \frac{\partial^j f}{\partial z_1^j} (r \zeta) \right|^p \, d\sigma(\zeta) \right)^{1/p} \lesssim \sum_{j=1}^{m+1} \left( \int_{S^n} \left( \int_0^1 \frac{1}{(1 + t - r \zeta_1)^{n/p-m+j}} \, dt \right)^p \, d\sigma(\zeta) \right)^{1/p} \lesssim \sum_{j=1}^{m+1} \int_0^1 \left( \int_{S^n} \frac{1}{|1 + t - r \zeta_1|^{n-mp+jp}} \, d\sigma(\zeta) \right)^{1/p} \frac{1}{t^{1-\sigma}} \, dt.
\]
Some elementary calculations show that
\[
\int_{S^n} \frac{d\sigma(\zeta)}{|1 + t - r \zeta_1|^{n-mp+jp}} \lesssim \frac{1}{(1 + t)^{n-mp+jp}} \int_{S^n} \frac{d\sigma(\zeta)}{|1 - (r/(1 + t)) \zeta_1, \zeta|^{n-mp+jp}} \lesssim \frac{1}{(1 + t)^{n-mp+jp}} \frac{1}{(1 - r/(1 + t))^p} \lesssim \frac{1}{(1 + t - r)^p},
\]
where \( \vec{e}_1 = (1, 0, \ldots, 0) \) and \( j = 1, \ldots, m + 1 \). Thus we have

\[
M_p(r, R^{m+1}f) \lesssim \int_0^1 \frac{1}{t^{1-\sigma}(1+t-r)} \, dt = \int_0^{1-r} \frac{1}{t^{1-\sigma}(1+t-r)} \, dt + \int_{1-r}^1 \frac{1}{t^{1-\sigma}(1+t-r)} \, dt.
\]

For the first integral we have

\[
\int_0^{1-r} \frac{1}{t^{1-\sigma}(1+t-r)} \, dt \leq \frac{1}{1-r} \int_0^{1-r} \frac{1}{t^{1-\sigma}} \, dt \lesssim \frac{1}{(1-r)^{1-\sigma}},
\]

and for the second integral we have

\[
\int_{1-r}^1 \frac{1}{t^{1-\sigma}(1+t-r)} \, dt \lesssim \int_{1-r}^1 \frac{1}{t^{2-\sigma}} \, dt \lesssim \frac{1}{(1-r)^{1-\sigma}}.
\]

Combining (5.2) and (5.3) we obtain

\[
M_p(r, R^{m+1}f) \lesssim \frac{1}{(1-r)^{1-\sigma}},
\]

which implies that \( f \in \Lambda^p_\alpha \).

On the other hand, we will show that \( f \notin H^p_\alpha \). To this end, we first use Lemma 5.4 to obtain

\[
\int_0^1 |R^\alpha f(rx)|^p(1-x)^n-1 \, dx \lesssim M_p(r, R^\alpha f).
\]

Observe that

\[
R^\alpha f(rx) = \frac{1}{\Gamma(1-\sigma)} \int_0^1 \left( \log \frac{1}{t} \right)^{-\sigma} R^{m+1}f(trx) \, \frac{dt}{t}.
\]

It follows that

\[
R^{m+1}f(trx) = \sum_{j=1}^{m+1} c_j z_j^1 \left( \frac{partial}{partial z_j^1} \right)(trx)
\]

\[
\gtrsim (trx)^{m+1} \frac{partial^{m+1}}{partial z_j^1}(trx)
\]

\[
= (trx)^{m+1} \int_0^1 \frac{1}{\rho^{1-\sigma}(1+\rho-trx)^{n/p+1}} \, d\rho
\]

\[
\gtrsim (trx)^{m+1} \frac{1}{(1-trx)^{n/p+1-\sigma}}.
\]

It is elementary to check that

\[
1 - t < \log \frac{1}{t} < 2(1-t), \quad 1/2 < t < 1.
\]
We may assume $rx > 1/2$. Then $1 - rx \sim 1 - trx$ for $rx < t < 1$, so
\[
R^\alpha f(rx) \gtrsim \int_{1/2}^1 \left(\log \frac{1}{t}\right)^{-\sigma} (trx)^{m+1} \left(1 -trx\right)^{n/p+1-\sigma} dt \\
\gtrsim (rx)^{2(m+1)} \int_{rx}^1 \frac{1}{(1-t)^\sigma (1-trx)^{n/p+1-\sigma}} dt \\
\gtrsim (rx)^{2(m+1)} \frac{1}{(1-rx)^{n/p+1-\sigma}} \int_{rx}^1 \frac{1}{(1-t)^\sigma} dt \\
\sim (rx)^{2(m+1)} \frac{1}{(1-rx)^{n/p}}.
\]
Therefore,
\[
\int_0^1 |R^\alpha f(rx)|^p (1-x)^{n-1} dx \gtrsim \int_{1/2}^1 (rx)^{2p(m+1)} \frac{1}{(1-rx)^n} dx \\
\gtrsim \int_{1/2}^1 \frac{1}{1-rx} dx \to \infty \quad \text{as} \quad r \to 1.
\]
This shows that $R^\alpha f \notin H^p$. 

\textbf{REFERENCES}


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