A SHARP NORM ESTIMATE OF THE BERGMAN PROJECTION ON L^p SPACES

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ABSTRACT. We show that the norm of the Bergman projection on L^p of the unit ball in $\mathbb{C}^n$ is comparable to $\csc(\pi/p)$ for $1 < p < \infty$.

1. INTRODUCTION

Throughout the paper we fix a positive integer n and let $\mathbb{B}$ denote the open unit ball in $\mathbb{C}^n$. For $-1 < \alpha < \infty$ let

$$dv_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dv(z),$$

where dv is the normalized volume measure on $\mathbb{B}$.

For $0 < p < \infty$ let

$$A^p_\alpha = H(\mathbb{B}) \cap L^p(\mathbb{B}, dv_\alpha)$$

denote the weighted Bergman space on $\mathbb{B}$ with standard radial weights. Here $H(\mathbb{B})$ is the space of all holomorphic functions in $\mathbb{B}$. It is easy to see that A^p_α is closed in $L^p(\mathbb{B}, dv_\alpha)$. We will use $\| \, \|_{p,\alpha}$ for the norm in $L^p(\mathbb{B}, dv_\alpha)$.

We use P_α to denote the orthogonal projection from $L^2(\mathbb{B}, dv_\alpha)$ onto A^2_α. It is well known that P_α is an integral operator on $L^2(\mathbb{B}, dv_\alpha)$,

$$P_\alpha f(z) = \int_{\mathbb{B}} K_\alpha(z,w) f(w) \, dv_\alpha(w),$$

where the integral kernel is given by

$$K_\alpha(z,w) = \frac{1}{(1-\langle z, w\rangle)^{n+1+\alpha}}.$$

It is also well known that, for $1 < p < \infty$, the Bergman projection P_α maps $L^p(\mathbb{B}, dv_\alpha)$ boundedly onto A^p_α. See Section 7.1 of [2] for example.

The purpose of this paper is to give a sharp estimate of the norm of P_α on $L^p(\mathbb{B}, dv_\alpha)$. Our main result is the following theorem.

Theorem. *For any $-1 < \alpha < \infty$ there exists a constant $C > 0$, depending on α and n but not on p, such that the norm of the operator*

$$P_\alpha : L^p(\mathbb{B}, dv_\alpha) \to A^p_\alpha$$

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satisfies the estimate

$$C^{-1}\csc\frac{\pi}{p}\le \|P_\alpha\|_p\le C\csc\frac{\pi}{p}$$

for all $1<p<\infty$.

It is easy to see that the quantity $\csc(\pi/p)$ is comparable to p as $p\to\infty$ and comparable $1/(p-1)$ as $p\to 1$.

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2. PRELIMINARIES

The proof of our main result still depends on the two traditional tools, namely, Forelli-Rudin type estimates for certain integrals on the ball and Schur's test for the boundedness of integral operators on L^p spaces. But three new ingredients are necessary here. First, we need a more precise version of the Forelli-Rudin estimates, namely, how the estimates depend on various parameters. Second, we need to find the right test function for Schur's lemma in order to control the parameters in the Forelli-Rudin estimates. And finally, we need to show that our estimates are sharp in a certain sense.

Lemma 1. *For any* $T>0$ *there exists a constant* $C>0$*, depending on* n *and* T *but not on* t*, such that*

$$\int_{\mathbb{S}}\frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+t}}\le\frac{C\Gamma(t)}{(1-|z|^2)^t}$$

for all $z\in\mathbb{B}$ *and* $0<t<T$*, where* $\mathbb{S}$ *is the unit sphere in* $\mathbb{C}^n$ *and* σ *is the normalized Lebesgue measure on* $\mathbb{S}$.

Proof. By the proof of Proposition 1.4.10 in [2],

$$\int_{\mathbb{S}}\frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+t}}=\frac{\Gamma(n)}{\Gamma^2(\lambda)}\sum_{k=0}^\infty\frac{\Gamma^2(k+\lambda)}{\Gamma(k+1)\Gamma(k+n)}|z|^{2k}, \tag{1}$$

where $\lambda=(n+t)/2$. Also,

$$\frac{1}{(1-|z|^2)^t}=\sum_{k=0}^\infty\frac{\Gamma(k+t)}{\Gamma(k+1)\Gamma(t)}|z|^{2k}. \tag{2}$$

As t goes from 0 to T, the parameter λ goes from $n/2$ to $(n+T)/2$, so the quotient $\Gamma(n)/\Gamma^2(\lambda)$ is bounded below away from 0 and bounded above away from infinity. We now use Stirling's formula to compare (1) and (2).

For each non-negative k let

$$a_k = a_k(t) = \frac{\Gamma^2(k+\lambda)}{\Gamma(k+n)\Gamma(k+t)}.$$

It is obvious that if $k > 0$ then a_k is positive and bounded in $t \in (0, T)$; this is also true when $k = 0$, because the gamma function is bounded away from 0 on the interval $(0, \infty)$. We need to show that there exists a positive constant C (depending only on n and T) such that $a_k \leq C$ for all $k \geq 0$ and $0 < t < T$.

By Stirling's formula, there exist positive constants C_1 and M such that

$$C_1^{-1} \leq \frac{\Gamma(x)}{x^{x-\frac{1}{2}}e^{-x}} \leq C_1$$

for all $x \geq M$. It then follows easily that there exists a constant $C_2 > 0$, depending only on n and T, such that

$$a_k \leq C_2 \left[1 + \frac{\lambda - n}{k+n}\right]^{k+n} \left[1 + \frac{\lambda - t}{k+t}\right]^{k+t} \frac{\sqrt{(k+n)(k+t)}}{k+\lambda}.$$

As $k \to +\infty$, the right hand side above approaches C_2 uniformly for $t \in (0, T)$ (when n is fixed), because

$$\lim_{y\to\infty} \left(1 + \frac{x}{y}\right)^y = e^x,$$

and the convergence is uniform for x in any finite interval. This proves the desired estimate. □

Lemma 2. *Given any $T > 0$ and $A > -1$, there exists a constant $C > 0$, depending on n, T, and A, but not on t and α, such that*

$$\int_{\mathbb{B}} \frac{(1-|w|^2)^\alpha \, dv(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+t}} \leq \frac{C\Gamma(\alpha+1)\Gamma(t)}{(1-|z|^2)^t}$$

for all $-1 < \alpha < A$, $0 < t < T$, and $z \in \mathbb{B}$.

Proof. Let I denote the integral concerned. By the proof of Proposition 1.4.10 in [2],

$$I = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\lambda)}{\Gamma(k+1)\Gamma(n+1+\alpha+k)} |z|^{2k}, \tag{3}$$

where $\lambda = (n+1+\alpha+t)/2$. The desired result then follows from (2), (3), and Stirling's formula. The details are exactly the same as in the proof of Lemma 1. □

Lemma 3. *Suppose μ is a positive measure on a space X and $H(x,y)$ is a positive kernel on X. If there exists a constant $C > 0$ and a positive function $h(x)$ on X such that*

$$\int_X H(x,y)h(y)^q d\mu(y) \le Ch(x)^q$$

for all x in X, and

$$\int_X H(x,y)h(x)^p d\mu(x) \le Ch(y)^p$$

for all y in X, then the integral operator

$$Tf(x) = \int_X H(x,y)f(y)\,d\mu(y)$$

is bounded on $L^p(X,\mu)$ with norm not exceeding C. Here $1 < p < \infty$ and $1/p + 1/q = 1$.

Proof. See [3] for example. □

3. PROOF OF THE MAIN RESULT

We now prove the main result of the paper.

Theorem 4. *For any $-1 < \alpha < \infty$ there exists a constant $C > 0$, depending on α and n but not on p, such that the norm of the operator*

$$P_\alpha : L^p(\mathbb{B}, dv_\alpha) \to A^p_\alpha$$

satisfies the estimate

$$C^{-1} \csc \frac{\pi}{p} \le \|P_\alpha\|_p \le C \csc \frac{\pi}{p}$$

for all $1 < p < \infty$.

Proof. Fix $1 < p < \infty$ and let q be the conjugate exponent,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Consider the function

$$h(z) = (1 - |z|^2)^{-(\alpha+1)/(pq)}$$

on $\mathbb{B}$ and the operator

$$Tf(z) = \int_{\mathbb{B}} \frac{f(w)\,dv_\alpha(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha}}$$

on $L^p(\mathbb{B}, dv_\alpha)$. By Lemma 2, with $T = \alpha + 1$ and $A = \alpha$, there exists a constant $C > 0$, independent of p, such that

$$\begin{aligned}\int_{\mathbb{B}} \frac{h(w)^q dv_\alpha(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha}} &= (\alpha+1)\int_{\mathbb{B}} \frac{(1-|w|^2)^{\frac{\alpha+1}{q}-1}\, dv(w)}{|1-\langle z, w\rangle|^{n+1+\frac{\alpha+1}{q}-1+\frac{\alpha+1}{p}}} \\ &\le \frac{C(\alpha+1)\Gamma\left(\frac{\alpha+1}{q}\right)\Gamma\left(\frac{\alpha+1}{p}\right)}{(1-|z|^2)^{(\alpha+1)/p}} \\ &= C(\alpha+1)\Gamma\left(\frac{\alpha+1}{q}\right)\Gamma\left(\frac{\alpha+1}{p}\right) h(z)^q.\end{aligned}$$

Similarly,

$$\int_{\mathbb{B}} \frac{h(z)^p dv_\alpha(z)}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \le C(\alpha+1)\Gamma\left(\frac{\alpha+1}{p}\right)\Gamma\left(\frac{\alpha+1}{q}\right) h(w)^p.$$

It follows from Lemma 3 that the norm of the operator T on $L^p(\mathbb{B}, dv_\alpha)$, and hence the norm of P_α on $L^p(\mathbb{B}, dv_\alpha)$, does not exceed

$$C(\alpha+1)\Gamma\left(\frac{\alpha+1}{q}\right)\Gamma\left(\frac{\alpha+1}{p}\right).$$

If $\alpha = 0$, a well-known property of the gamma function gives

$$\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}\right) = \frac{\pi}{\sin\frac{\pi}{p}};$$

see [1].

If $\alpha \neq 0$, we can find a constant $C_1 > 0$, independent of p but dependent on α, such that

$$\Gamma\left(\frac{\alpha+1}{q}\right)\Gamma\left(\frac{\alpha+1}{p}\right) \le \frac{C_1}{\sin\frac{\pi}{p}}.$$

In fact, because of the symmetry of the sine function and the conjugacy between p and q, we only need to consider the case in which p is very large. In this case, the factor $\Gamma((\alpha+1)/q)$ is bounded from above and from below, and

$$\Gamma\left(\frac{\alpha+1}{p}\right) \sim p \sim \frac{1}{\sin\frac{\pi}{p}},$$

because

$$x\Gamma(x) = \Gamma(x+1) \sim 1$$

when x is a small positive number.

To prove that the norm estimate

$$\|P_\alpha\|_p \le \frac{C}{\sin\frac{\pi}{p}}$$

is sharp, we only need to consider the case when $p > 2$; the case when $1 < p < 2$ then follows from duality and the symmetry of the function $\sin(\pi/p)$. Note again that for $p > 2$ the constant $\sin(\pi/p)$ is comparable to $1/p$.

So we assume that $p > 2$ and consider the function

$$\begin{aligned} f(z) &= \log(1-z_1) - \overline{\log(1-z_1)} \\ &= 2i\arg(1-z_1). \end{aligned}$$

It is clear that $|f(z)| \leq 2\pi$ for all $z \in \mathbb{B}$, so that the norm of f in $L^p(\mathbb{B}, dv_\alpha)$ does not exceed 2π. On the other hand, it is easy to see that

$$P_\alpha f(z) = \log(1-z_1);$$

see Theorem 7.1.4(b) in [2].

It is well known that every function g in A^p_α satisfies the pointwise estimate

$$|g(z)| \leq \frac{\|g\|_{p,\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}}, \qquad z \in \mathbb{B}. \tag{4}$$

In fact, for any fixed z in $\mathbb{B}$, a change of variables shows that the function g and G have the same norm in A^p_α, where

$$G(w) = g \circ \varphi_z(w) \left[\frac{(1-|z|^2)^{n+1+\alpha}}{(1-\langle w, z\rangle)^{2(n+1+\alpha)}} \right]^{\frac{1}{p}}, \qquad w \in \mathbb{B},$$

and φ_z is the involutive automorphism of $\mathbb{B}$ that interchanges the orgin and the point z; see Section 2.2 in [2]. The obvious estimate $|G(0)| \leq \|G\|_{p,\alpha}$ then leads to (4).

Let $g = P_\alpha f$ and $z = (r, 0, \cdots, 0)$ in (4), where $0 < r < 1$. We obtain

$$\begin{aligned} \|P_\alpha f\|_{p,\alpha} &\geq (1-r^2)^{\frac{n+1+\alpha}{p}} \log \frac{1}{1-r} \\ &\geq (1-r)^{\frac{n+1+\alpha}{p}} \log \frac{1}{1-r}. \end{aligned}$$

In particular, if $r = 1 - e^{-p}$, then

$$\|P_\alpha f\|_{p,\alpha} \geq p e^{-(n+1+\alpha)}.$$

This shows that

$$\frac{\|P_\alpha f\|_{p,\alpha}}{\|f\|_{p,\alpha}} \geq \frac{p}{2\pi e^{n+1+\alpha}},$$

so the norm of P_α on $L^p(\mathbb{B}, dv_\alpha)$ is at least $p/(2\pi e^{n+1+\alpha})$. This completes the proof of the theorem. □

Note that the main result can be restated as follows: there exists a constant $C > 0$ such that

$$C^{-1}\frac{p^2}{p-1} \le \|P_\alpha\|_p \le C\frac{p^2}{p-1}$$

for every $p \in (1, \infty)$. The quotient $p^2/(p-1)$ can also be replaced by pq or $p+q$, where $1/p + 1/q = 1$.

4. Related Questions

Our main result shows how fast the norm of the Bergman projection P_α on $L^p(\mathbb{B}, dv_\alpha)$ grows as p increases to infinity or as p decreases to 1, when α is fixed. A related question is to determine how the norm of P_α on $L^p(\mathbb{B}, dv_\alpha)$ depends on α when p is fixed. In particular, we are interested in estimates of this norm when p is fixed and when α approaches -1. We conjecture that the norm of P_α on $L^p(\mathbb{B}, dv_\alpha)$ remains bounded if p is fixed in $(1, \infty)$ and when α approaches -1. A direct proof of this, such as the one in the previous section, will give a proof for the boundedness of the Cauchy-Szëgo projection on L^p spaces of the unit sphere when $1 < p < \infty$.

It is of course well known that the Cauchy-Szëgo projection Q is bounded on L^p spaces of the unit sphere when $1 < p < \infty$. However, we are not aware of any estimates for the norm of Q on L^p.

Another natural problem is to find sharp norm estimates for the Bergman projection on L^p spaces of other domains, such as strongly pseudo-convex domains in $\mathbb{C}^n$.

References

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