NEW CHARACTERIZATIONS OF BERGMAN SPACES

MIROSLAV PAVLOVIĆ AND KEHE ZHU

ABSTRACT. We obtain several new characterizations for the standard weighted Bergman spaces A^p_α on the unit ball of $\mathbb{C}^n$ in terms of the radial derivative, the holomorphic gradient, and the invariant gradient.

1. INTRODUCTION

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$. For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha\, dv(z),$$

where dv is the normalized volume measure on $\mathbb{B}_n$ and c_α is a positive constant making dv_α a probability measure. For $0 < p < \infty$ the weighted Bergman space A^p_α consists of holomorphic functions in $L^p(\mathbb{B}_n, dv_\alpha)$. Thus

$$A^p_\alpha = H(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, dv_\alpha),$$

where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$.

For $f \in H(\mathbb{B}_n)$ and $z = (z_1, \cdots, z_n) \in \mathbb{B}_n$ we define

$$Rf(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$$

and call it the radial derivative of f at z. The complex gradient of f at z is defined as

$$|\nabla f(z)| = \left[\sum_{k=1}^n \left|\frac{\partial f}{\partial z_k}(z)\right|^2\right]^{1/2}.$$

Let $\mathrm{Aut}(\mathbb{B}_n)$ denote the automorphism group of $\mathbb{B}_n$. Thus $\mathrm{Aut}(\mathbb{B}_n)$ consists of all bijective holomorphic functions $\varphi : \mathbb{B}_n \to \mathbb{B}_n$. It is well known that $\mathrm{Aut}(\mathbb{B}_n)$ is generated by two types of maps: unitaries and symmetries. The unitaries are simiply the $n \times n$ unitary matrices considered as mappings from $\mathbb{B}_n$ to $\mathbb{B}_n$. For any point $a \in \mathbb{B}_n$ there exists a unique map

Date: January 30, 2006.

2000 *Mathematics Subject Classification.* Primary 32A36, secondary 46E20.

Key words and phrases. Bergman spaces, unit ball, radial derivative, gradient, invariant gradient.

The first author is supported in part by MNZZS Grant ON144010 and the second author is partially supported by the National Science Foundation.

$\varphi_a \in \mathrm{Aut}(\mathbb{B}_n)$ with the following properties: $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a \circ \varphi_a(z) = z$ for all $z \in \mathbb{D}$. Such a mapping φ_a is called a symmetry. Because of the property $\varphi_a \circ \varphi_a(z) = z$ it is also natural to call φ_a an involution or an involutive automorphism. See [2] and [3] for more information about the automorphism group of $\mathbb{B}_n$.

If $f \in H(\mathbb{B}_n)$, we define

$$|\widetilde{\nabla} f(z)| = |\nabla(f \circ \varphi_z)(0)|, \qquad z \in \mathbb{B}_n.$$

It can be checked that

$$|\widetilde{\nabla}(f \circ \varphi)| = |(\widetilde{\nabla} f) \circ \varphi|, \qquad \varphi \in \mathrm{Aut}(\mathbb{B}_n).$$

So $|\widetilde{\nabla} f(z)|$ is called the invariant gradient of f at z. See [3] for more information about the invariant gradient.

When $n = 1$, the unit ball $\mathbb{B}_1$ is usually called the unit disk and we denote it by $\mathbb{D}$ instead. In this case, we clearly have

$$Rf(z) = zf(z), \quad |\nabla f(z)| = |f'(z)|, \quad |\widetilde{\nabla} f(z)| = (1 - |z|^2)|f'(z)|.$$

In particular, the functions

$$(1 - |z|^2)|Rf(z)|, \quad (1 - |z|^2)|\nabla f(z)|, \quad |\widetilde{\nabla} f(z)|, \tag{1}$$

have exactly the same boundary behavior on the unit disk $\mathbb{D}$. In higher dimensions, the three functions above no longer have the same boundary behavior; see Section 2.3 and Chapter 7 in [3]. However, when integrated against the weighted volume measures dv_α, not only do these differential-based functions exhibit the same behavior, they also behave the same as the original function $f(z)$, as the following result (see Theorem 2.16 of [3]) demonstrates.

Theorem 1. *Suppose $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

(a) *$f \in A^p_\alpha$, that is, $f \in L^p(\mathbb{B}_n, dv_\alpha)$.*
(b) *The function $f_1(z) = (1 - |z|^2)|Rf(z)|$ belongs to $L^p(\mathbb{B}_n, dv_\alpha)$.*
(c) *The function $f_2(z) = (1 - |z|^2)|\nabla f(z)|$ belongs to $L^p(\mathbb{B}_n, dv_\alpha)$.*
(d) *The function $f_3(z) = |\widetilde{\nabla} f(z)|$ belongs to $L^p(\mathbb{B}_n, dv_\alpha)$.*

Moreover, the quantities

$$|f(0)|^p + \int_{\mathbb{B}_n} |f_1|^p \, dv_\alpha, \; |f(0)|^p + \int_{\mathbb{B}_n} |f_2|^p \, dv_\alpha, \; |f(0)|^p + \int_{\mathbb{B}_n} |f_3|^p \, dv_\alpha,$$

are all comparable to

$$\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)$$

whenever f is holomorphic in $\mathbb{B}_n$.

The purpose of this paper is to explore the above ideas further. We show that the integral behavior of the functions

$$|f(z)|, \quad (1-|z|^2)|Rf(z)|, \quad (1-|z|^2)|\nabla f(z)|, \quad |\widetilde{\nabla} f(z)|,$$

is the same in a much stronger sense. More specifically, when integrating over the unit ball with respect to weighted volume measures, we can write $|f(z)|^p = |f(z)|^{p-q}|f(z)|^q$ and can replace $|f(z)|$ in the second factor by any one of the functions in (1). We state our main result as follows.

Theorem 2. *Suppose $p > 0$, $\alpha > -1$, $0 < q < p+2$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

(a) $f \in A^p_\alpha$, *that is,* $I_1(f) < \infty$, *where*

$$I_1(f) = \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z).$$

(b) $I_2(f) < \infty$, *where*

$$I_2(f) = \int_{\mathbb{B}_n} |f(z)|^{p-q} \left[(1-|z|^2)|Rf(z)|\right]^q \, dv_\alpha(z).$$

(c) $I_3(f) < \infty$, *where*

$$I_3(f) = \int_{\mathbb{B}_n} |f(z)|^{p-q} \left[(1-|z|^2)|\nabla f(z)|\right]^q \, dv_\alpha(z).$$

(d) $I_4(f) < \infty$, *where*

$$I_4(f) = \int_{\mathbb{B}_n} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q \, dv_\alpha(z).$$

Furthermore, the quantities

$$I_1(f), \quad |f(0)|^p + I_2(f), \quad |f(0)|^p + I_3(f), \quad |f(0)|^p + I_4(f),$$

are comparable for $f \in H(\mathbb{B}_n)$.

We will show by a simple example that the range $0 < q < p+2$ is best possible.

Throughout the paper we use C to denote a positive constant, indepedent of f and z, whose value may vary from one occurence to another.

2. The case $0 < q \le p$

The proof of Theorem 2 requires different methods for the two cases $0 < q \le p$ and $p < q < p+2$. This section deals with the case $0 < q \le p$; the other case is considered in the next section.

The case $q = p$ is of course just Theorem 1. Our proof of Theorem 2 in the case $0 < q < p$ is based on several technical lemmas that are known to experts. We include them here for the non-expert and for convenience of reference. We begin with the following embedding theorem for Bergman spaces.

Lemma 3. *Suppose* $0 < p \le 1$, $\alpha > -1$, *and*

$$\beta = \frac{n+1+\alpha}{p} - (n+1).$$

There exists a constant $C > 0$ *such that*

$$\int_{\mathbb{B}_n} |f(z)|\, dv_\beta(z) \le C \left[\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)\right]^{1/p}$$

for all $f \in H(\mathbb{B}_n)$.

Proof. See Lemma 2.15 of [3]. □

We will also need the following boundedness criterion for a class of integral operators on $\mathbb{B}_n$.

Lemma 4. *For real* a *and* b *consider the integral operator* $T = T_{a,b}$ *defined by*

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{|1-\langle z, w\rangle|^{n+1+a+b}} f(w)\, dv(w),$$

where

$$\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$$

for $z = (z_1, \cdots, z_n)$ *and* $w = (w_1, \cdots, w_n)$ *in* $\mathbb{B}_n$. *If* $p \ge 1$, *then* T *is bounded on* $L^p(\mathbb{B}_n, dv_\alpha)$ *if and only if the inequalities*

$$-pa < \alpha + 1 < p(b+1)$$

hold.

Proof. See Theorem 2.10 of [3]. □

The following result compares the various derivatives that we use for a holomorphic function in $\mathbb{B}_n$.

Lemma 5. *If* $f \in H(\mathbb{B}_n)$, *then*

$$|\widetilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2).$$

Moreover,

$$(1-|z|^2)|Rf(z)| \le (1-|z|^2)|\nabla f(z)| \le |\widetilde{\nabla} f(z)|$$

for all $z \in \mathbb{B}_n$.

Proof. See Lemmas 2.13 and 2.14 of [3]. □

We will need the following well-known reproducing formula for holomorphic functions in $\mathbb{B}_n$.

Lemma 6. *If* $\alpha > -1$ *and* $f \in A^1_\alpha$, *then*

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)\, dv_\alpha(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}$$

for all $z \in \mathbb{B}_n$.

Proof. See Theorem 2.2 of [3]. □

The following integral estimate is standard in the theory of Bergman spaces and has proved to be very useful in many different situations.

Lemma 7. *Suppose* $\alpha > -1$ *and* $t > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{B}_n} \frac{dv_\alpha(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha+t}} \le \frac{C}{(1 - |z|^2)^t}$$

for all $z \in \mathbb{B}_n$.

Proof. See Proposition 1.4.10 of [2] or Theorem 1.12 of [3]. □

We now begin the proof of Theorem 2 under the assumption that $0 < q < p$. In this case, the numbers $r = p/(p-q)$ and $s = p/q$ satisfy $r > 1$, $s > 1$, and $1/r + 1/s = 1$. So we can apply Hölder's inequality to the integral $I_4(f)$ to obtain

$$I_4(f) \le \left[\int_{\mathbb{B}_n} |f(z)|^p \, dA_\alpha(z)\right]^{\frac{1}{r}} \left[\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \, dv_\alpha(z)\right]^{\frac{1}{s}}. \qquad (2)$$

By Theorem 1, there exists a positive constant $C > 0$, independent of f, such that

$$\int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p \, dv_\alpha(z) \le C \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z).$$

Combining this with (2), we see that the integral $I_4(f)$ is dominated by $I_1(f)$.

According to Lemma 5, we have $I_2(f) \le I_3(f) \le I_4(f)$. So it remains for us to show that $I_1(f)$ is finite whenever $I_2(f)$ is finite. We do this in two steps.

First, we assume that $p = qN$ for some integer $N > 1$. In this case, the function $f(z)^{p/q}$ is well-defined and holomorphic in $\mathbb{B}_n$. Moreover,

$$R\left[f(z)^{\frac{p}{q}}\right] = \frac{p}{q} f(z)^{\frac{p}{q}-1} Rf(z).$$

Let β be a sufficiently large (to be specified later) positive integer and apply Lemma 6 to write

$$R\left[f(z)^{\frac{p}{q}}\right]=\frac{p}{q}\int_{\mathbb{B}_n}\frac{f(w)^{\frac{p}{q}-1}Rf(w)\,dv_\beta(w)}{(1-\langle z,w\rangle)^{n+1+\beta}},\qquad z\in\mathbb{B}_n.$$

Since the function $f(w)^{(p/q)-1}Rf(w)$ vanishes at the origin, we can also write

$$R\left[f(z)^{\frac{p}{q}}\right]=\frac{p}{q}\int_{\mathbb{B}_n}\left[\frac{1}{(1-\langle z,w\rangle)^{n+1+\beta}}-1\right]f(w)^{\frac{p}{q}-1}Rf(w)\,dv_\beta(w).$$

Integrating the above equation, we obtain

$$f(z)^{\frac{p}{q}}-f(0)^{\frac{p}{q}}=\int_0^1 Rf^{\frac{p}{q}}(tz)\,\frac{dt}{t}=\int_{\mathbb{B}_n}H(z,w)f(w)^{\frac{p}{q}-1}Rf(w)\,dv_\beta(w),$$

where

$$H(z,w)=\frac{p}{q}\int_0^1\frac{1-(1-t\langle z,w\rangle)^{n+1+\beta}}{(1-t\langle z,w\rangle)^{n+1+\beta}}\,\frac{dt}{t}.$$

Expand the numerator in the integrand above by the binomial formula and then evaluate the integral term by term. We obtain a positive constant $C>0$ such that

$$|H(z,w)|\le\frac{C}{|1-\langle z,w\rangle|^{n+\beta}}$$

for all z and w in $\mathbb{B}_n$. It follows that

$$\left|f(z)^{\frac{p}{q}}-f(0)^{\frac{p}{q}}\right|\le C\int_{\mathbb{B}_n}\frac{|f(w)|^{\frac{p}{q}-1}|Rf(w)|\,dv_\beta(w)}{|1-\langle z,w\rangle|^{n+\beta}}\tag{3}$$

for all $z\in\mathbb{B}_n$.

If $q\ge1$, then we rewrite (3) as

$$\left|f(z)^{\frac{p}{q}}-f(0)^{\frac{p}{q}}\right|\le C\int_{\mathbb{B}_n}g(w)\frac{(1-|w|^2)^{\beta-1}\,dv(w)}{|1-\langle z,w\rangle|^{n+1+\beta-1}},\tag{4}$$

where

$$g(w)=|f(w)|^{\frac{p}{q}-1}(1-|w|^2)|Rf(w)|.$$

By Lemma 4, the integral operator

$$Tg(z)=\int_{\mathbb{B}_n}g(w)\,\frac{(1-|w|^2)^{\beta-1}\,dv(w)}{|1-\langle z,w\rangle|^{n+1+\beta-1}}$$

is bounded on $L^q(\mathbb{B}_n,dv_\alpha)$, because we can choose the positive integer β to satisfy $\alpha+1<q\beta$. Combining this with (4), we obtain a positive constant C, independent of f, such that

$$\int_{\mathbb{B}_n}\left|f^{\frac{p}{q}}-f(0)^{\frac{p}{q}}\right|^q dv_\alpha\le C\int_{\mathbb{B}_n}|f(z)|^{p-q}\left[(1-|z|^2)|Rf(z)|\right]^q dv_\alpha(z).$$

This clearly shows that there exists a positive constant $C > 0$, independent of f, such that

$$I_1(f) \le C\left[|f(0)|^p + I_2(f)\right]$$

for all $f \in H(\mathbb{B}_n)$.

If $0 < q < 1$, we rewrite (3) as

$$\left|f(z)^{\frac{p}{q}} - f(0)^{\frac{p}{q}}\right| \le C\int_{\mathbb{B}_n}\left|\frac{f(w)^{\frac{p}{q}-1}Rf(w)}{(1-\langle w,z\rangle)^{n+\beta}}\right|(1-|w|^2)^\beta\,dv(w). \qquad (5)$$

We also write

$$\beta = \frac{n+1+\gamma}{q} - (n+1),$$

and choose β to be large enough so that $\gamma > -1$. We then apply Lemma 3 to the right-hand side of (5) to obtain

$$\left|f(z)^{\frac{p}{q}} - f(0)^{\frac{p}{q}}\right| \le C\left[\int_{\mathbb{B}_n}\left|\frac{f(w)^{\frac{p}{q}-1}Rf(w)}{(1-\langle z,w\rangle)^{n+\beta}}\right|^q dv_\gamma(w)\right]^{\frac{1}{q}},$$

where C is a positive constant independent of f. Take the qth power on both sides, integrate over $\mathbb{B}_n$ with respect to dv_α, and apply Fubini's theorem. We see that the integral

$$\int_{\mathbb{B}_n}\left|f(z)^{\frac{p}{q}} - f(0)^{\frac{p}{q}}\right|^q dv_\alpha$$

is dominated by the integral

$$\int_{\mathbb{B}_n}|f(w)|^{p-q}|Rf(w)|^q\,dv_\gamma(w)\int_{\mathbb{B}_n}\frac{dv_\alpha(z)}{|1-\langle z,w\rangle|^{q(n+\beta)}}.$$

If β is large enough so that

$$q(n+\beta) > n+1+\alpha,$$

then by Lemma 7, there exists a positive constant C such that

$$\int_{\mathbb{B}_n}\frac{dv_\alpha(z)}{|1-\langle z,w\rangle|^{q(n+\beta)}} \le \frac{C}{(1-|w|^2)^{q(n+\beta)-(n+1+\alpha)}}$$

for all $w \in \mathbb{B}_n$. An easy calculation shows that

$$q(n+\beta) - (n+1+\alpha) = \gamma - (q+\alpha).$$

It follows that

$$\int_{\mathbb{B}_n}\left|f^{\frac{p}{q}} - f(0)^{\frac{p}{q}}\right|^q dv_\alpha \le C\int_{\mathbb{B}_n}|f(z)|^{p-q}\left[(1-|z|^2)|Rf(z)|\right]^q dv_\alpha(z),$$

where C is a positive constant independent of f. This easily implies that

$$I_1(f) \le C\left[|f(0)|^p + I_2(f)\right]$$

for another positive constant C that is independent of f.

Thus we have proved that the integral $I_1(f)$ is dominated by $|f(0)|^p + I_2(f)$ under the additional assumption that $p = qN$, where $N > 1$ is a positive integer.

In the general case $0 < q < p$, we choose a positive integer N such that $Nq > p$ and define two positive numbers r and s by

$$r = \frac{Nq}{p}, \qquad \frac{1}{r} + \frac{1}{s} = 1.$$

By the special case that we have already proved, there exists a constant $C > 0$, independent of f, such that

$$I_1(f) \le C \left[|f(0)|^p + \int_{\mathbb{B}_n} \left[|f(z)|^{-1}(1-|z|^2)|Rf(z)| \right]^{p/N} |f(z)|^p \, dv_\alpha(z) \right].$$

By an approximation argument we may assume that $I_1(f)$ is finite (note that we are trying to prove the stronger conclusion that $I_1(f)$ is dominated by $|f(0)|^p + I_2(f)$). By Hölder's inequality, the integral on the right-hand side above does not exceed

$$\left[\int_{\mathbb{B}_n} \left[|f(z)|^{-1}(1-|z|^2)|Rf(z)| \right]^{rp/N} |f(z)|^p \, dv_\alpha(z) \right]^{\frac{1}{r}} \left[\int_{\mathbb{B}_n} |f|^p \, dv_\alpha \right]^{\frac{1}{s}}.$$

It follows that

$$I_1(f) \le C \left[|f(0)|^p + I_2(f)^{\frac{1}{r}} I_1(f)^{\frac{1}{s}} \right].$$

From this we easily deduce that $I_1(f)$ is dominated by $|f(0|^p + I_2(f)$. In fact, this is obvious if $f(0) = 0$. Otherwise, we may use homogeneity to assume that $f(0) = 1$. In this case, we also have $I_1(f) \ge 1$, so dividing both sides of the above inequality by $I_1(f)^{1/s}$ yields

$$I_1(f)^{\frac{1}{r}} \le C \left[\frac{1}{I_1(f)^{1/s}} + I_2(f)^{\frac{1}{r}} \right] \le C \left[1 + I_2(f)^{\frac{1}{r}} \right].$$

This clearly implies that

$$I_1(f) \le C\,[1 + I_2(f)] = C\,[|f(0)|^p + I_2(f)]$$

for some other positive constant independent of f. This completes the proof of Theorem 2 in the case $0 < q \le p$.

3. THE CASE $p < q < p + 2$

This section is devoted to the proof of Theorem 2 in the case $p < q < p + 2$.

It follows from Theorem 1 that there exists a small positive constant c such that

$$\begin{aligned} cI_1(f) - |f(0)|^p &\leq \int_{\mathbb{B}_n} (1-|z|^2)^p |Rf(z)|^p \, dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} (1-|z|^2)^p |Rf(z)|^p |f(z)|^a |f(z)|^{-a} \, dv_\alpha(z), \end{aligned}$$

where $a = p(p-q)/q$. Let

$$r = \frac{q}{p}, \qquad s = \frac{q}{q-p}.$$

When $p < q$, we have $r > 1$, $s > 1$, and $1/r + 1/s = 1$. An application of Hölder's inequality shows that $cI_1(f) - |f(0)|^p$ does not exceed

$$\left[\int_{\mathbb{B}_n} (1-|z|^2)^q |Rf(z)|^q |f(z)|^{p-q} \, dv_\alpha(z)\right]^{\frac{1}{r}} \left[\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)\right]^{\frac{1}{s}}.$$

Therefore,

$$cI_1(f) \leq |f(0)|^p + I_2(f)^{\frac{1}{r}} I_1(f)^{\frac{1}{s}}.$$

From this we easily deduce that

$$I_1(f) \leq C\left[|f(0)|^p + I_2(f)\right]$$

for some positive constant C independent of f; see the last paragraph of the previous section.

Once again, Lemma 5 tells us that $I_2(f) \leq I_3(f) \leq I_4(f)$. So it remains for us to show that the integral $I_4(f)$ is dominated by $I_1(f)$. This will require several technical lemmas again.

We begin with the following well-known estimate for the Bergman kernel on pseudo-hyperbolic balls.

Lemma 8. *Suppose* $\rho \in (0,1)$. *Then there exists a positive constant* C *(independent of* z *and* w*) such that*

$$C^{-1}(1-|z|^2) \leq |1-\langle z,w\rangle| \leq C(1-|w|^2)$$

for all z *and* w *in* $\mathbb{B}_n$ *satisfying* $|\varphi_z(w)| < \rho$. *Moreover, if*

$$D(z,\rho) = \{w \in \mathbb{B}_n : |\varphi_z(w)| < \rho\}$$

is a pseudo-hyperbolic ball, then its Euclidean volume satisfies

$$C^{-1}(1-|z|^2)^{n+1} \leq v(D(z,\rho)) \leq C(1-|z|^2)^{n+1}.$$

Proof. See Lemmas 1.23 and 2.20 of [3]. □

Note that, by symmetry, the positions of z and w can be interchanged in the first set of inequalities of Lemma 8.

The key to the remaining proof of Theorem 2 is the following well-known special case of $q = 2$.

Lemma 9. *For every $p > 0$ there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} |f(z)|^p\,dv(z) \le C\left[|f(0)|^p + \int_{\mathbb{B}_n} |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2\,dv(z)\right]$$

and

$$|f(0)|^p + \int_{\mathbb{B}_n} |f(z)|^{p-2}|\widetilde{\nabla} f(z)|^2\,dv(z) \le C\int_{\mathbb{B}_n} |f(z)|^p\,dv(z)$$

for all $f \in H(\mathbb{B}_n)$.

Proof. See [1]. □

In the general case, we first prove the following weaker version.

Lemma 10. *Suppose $p > 0$, $0 < q < p + 2$, and $\alpha > -1$. There exists a positive constant C (independent of f) such that*

$$\int_{|z|<1/4} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q\,dv_\alpha(z) \le C\int_{|z|<3/4} |f(z)|^p\,dv_\alpha(z)$$

for all $f \in H(\mathbb{B}_n)$.

Proof. If $0 < q \le p$, the desired estimate follows from the well-known fact that point-evaluations (of any form of the derivative) on a compact subset of $|z| < 3/4$ are uniformly bounded linear functionals on the Bergman spaces of the ball $|z| < 3/4$; see Lemma 2.4 of [3] for example.

So we assume that $p < q < p + 2$. In this case, we have $1 < 2/(q - p)$. Fix $r \in (1, 2/(q - p))$, sufficiently close to $2/(q - p)$, so that $q - \lambda > 0$, where $\lambda = 2/r \in (q - p, 2)$.

If f is a unit vector in $H^\infty(\mathbb{B}_n)$, then there exists a constant $C > 0$, independent of f, such that $|\nabla f(0)| \le C$. Replacing f by $f \circ \varphi_z$, we obtain $|\widetilde{\nabla} f(z)| \le C$ for all $z \in \mathbb{B}_n$. It follows from this and Hölder's inequality that the integral

$$I(f) = \int_{|z|<1/2} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q\,dv(z)$$

satisfies

$$\begin{aligned} I(f) &= \int_{|z|<1/2} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^{\lambda} |\widetilde{\nabla} f(z)|^{q-\lambda}\, dv(z) \\ &\le C^{q-\lambda} \int_{|z|<1/2} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^{\lambda}\, dv(z) \\ &\le C^{q-\lambda} \left[\int_{|z|<1/2} |f(z)|^{r(p-q)} |\widetilde{\nabla} f(z)|^{r\lambda}\, dv(z) \right]^{\frac{1}{r}} \\ &\le C^{q-\lambda} \left[\int_{\mathbb{B}_n} |f(z)|^{r(p-q)} |\widetilde{\nabla} f(z)|^{r\lambda}\, dv(z) \right]^{\frac{1}{r}} \\ &= C^{q-\lambda} \left[\int_{\mathbb{B}_n} |f(z)|^{r(p-q)+2-2} |\widetilde{\nabla} f(z)|^{2}\, dv(z) \right]^{\frac{1}{r}}. \end{aligned}$$

By Lemma 9, there exists a positive constant C, independent of f, such that

$$I(f) \le C \left[\int_{\mathbb{B}_n} |f(z)|^{r(p-q)+2}\, dv(z) \right]^{\frac{1}{r}} \le C$$

for all unit vectors f of $H^\infty(\mathbb{B}_n)$. Here we used the assumption that $r(p-q)+2>0$, which is equivalent to $r<2/(q-p)$. If f is an arbitrary function in $H^\infty(\mathbb{B}_n)$, then replacing f by $f/\|f\|_\infty$ in $I(f)\le C$ leads to

$$\int_{|z|<1/2} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^{q}\, dv(z) \le C\|f\|_\infty^p, \tag{6}$$

where

$$\|f\|_\infty = \sup\{|f(z)| : z\in\mathbb{B}_n\}.$$

It is easy to see that $|\widetilde{\nabla} f(z)|$ and $|\nabla f(z)|$ are comparable on any compact subset of $\mathbb{B}_n$. In fact, it follows from Lemma 5 that

$$(1-|z|^2)|\nabla f(z)| \le |\widetilde{\nabla} f(z)| \le |\nabla f(z)|,$$

which shows that $|\widetilde{\nabla} f(z)|$ and $|\nabla f(z)$ are comparable on any compact subset of $\mathbb{B}_n$.

Now suppose f is any holomorphic function in $\mathbb{B}_n$. We replace $f(z)$ in (6) by $f(z/2)$, use the conclusion of the previous paragraph, and make the change of variables $w=z/2$. Then there exists a positive constant C, independent of f, such that

$$\int_{|z|<1/4} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^{q}\, dv(z) \le C \sup\{|f(z)|^p : |z|\le 1/2\}.$$

Since point-evaluations in $|z|\le 1/2$ are uniformly bounded on Bergman spaces of the ball $|z|<3/4$, there exists a positive constant C, independent of f, such that

$$\int_{|z|<1/4} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q\, dv(z) \le C \int_{|z|<3/4} |f(z)|^p\, dv(z).$$

Since $(1-|z|^2)^\alpha$ is comparable to a positive constant whenever z is restricted to a compact subset of $\mathbb{B}_n$, we obtain a positive constant C, independent of f, such that

$$\int_{|z|<1/4} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q\, dv_\alpha(z) \le C \int_{|z|<3/4} |f(z)|^p\, dv_\alpha(z).$$

This completes the proof of Lemma 10. □

We now use Lemma 10 to show that the integral $I_4(f)$ is dominated by $I_1(f)$. This part of the proof works for the full range $0 < q < p+2$.

Replace f by $f\circ\varphi_w$ in Lemma 10, where w is an arbitrary point in $\mathbb{B}_n$, and use the Möbius invariance of $\widetilde{\nabla} f$. Then the integrals

$$\int_{|z|<1/4} |f(\varphi_w(z))|^{p-q}|(\widetilde{\nabla} f)(\varphi_w(z))|^q\, dv_\alpha(z)$$

are uniformly (with respecto to w) dominated by the integrals

$$\int_{|z|<3/4} |f(\varphi_w(z))|^p\, dv_\alpha(z).$$

Making the change of variables $z\mapsto\varphi_w(z)$ in the above integrals, we see that the integrals

$$\int_{|\varphi_w(z)|<1/4} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q \frac{(1-|w|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(z)$$

are uniformly (with respect to w) dominated by the integrals

$$\int_{|\varphi_w(z)|<3/4} |f(z)|^p \frac{(1-|w|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(z).$$

According to Lemma 8, for $|\varphi_w(z)| < 3/4$ (hence for $|\varphi_w(z)| < 1/4$ as well) we have

$$1-|w|^2 \sim 1-|z|^2 \sim |1-\langle z,w\rangle|.$$

It follows that there exists another positive constant C, independent of f and w, such that

$$\int_{|\varphi_w(z)|<1/4} |f(z)|^{p-q}|\widetilde{\nabla} f(z)|^q\, dv_\alpha(z) \le C \int_{|\varphi_w(z)|<3/4} |f(z)|^p\, dv_\alpha(z)$$

for all $f \in H(\mathbb{B}_n)$. Integrate the above inequality over $\mathbb{B}_n$ with respect to the Möbius invariant measure

$$d\tau(w) = \frac{dv(w)}{(1-|w|^2)^{n+1}}.$$

We see that the integral

$$\int_{\mathbb{B}_n} d\tau(w) \int_{|\varphi_z(w)|<1/4} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q \, dv_\alpha(z) \tag{7}$$

is dominated by the integral

$$\int_{\mathbb{B}_n} d\tau(w) \int_{|\varphi_z(w)|<3/4} |f(z)|^p \, dv_\alpha(z). \tag{8}$$

By Fubini's theorem, the integral in (7) equals

$$\int_{\mathbb{B}_n} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q \, dv_\alpha(z) \int_{|\varphi_w(z)|<1/4} d\tau(w).$$

Similarly, the integral in (8) equals

$$\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \int_{|\varphi_w(z)|<3/4} d\tau(w).$$

For any fixed radius $\rho \in (0,1)$, it follows from Lemma 8 that the integral

$$\int_{|\varphi_w(z)|<\rho} d\tau(w)$$

is comparable to a positive constant. Combining these conclusions with (7) and (8), we obtain another positive constant C, independent of f, such that

$$\int_{\mathbb{B}_n} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q \, dv_\alpha(z) \le C \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)$$

for all $f \in H(\mathbb{B}_n)$. This shows that the integral $I_4(f)$ is always dominated by $I_1(f)$. The proof of Theorem 2 is now complete.

4. Further Remarks

An immediate consequence of Theorem 2 is the following characterization of Bergman spaces in terms of the familiar first order partial derivatives.

Corollary 11. *Suppose* $p > 0$, $0 < q < p+2$, $\alpha > -1$, *and* f *is holomorphic in* $\mathbb{B}_n$. *Then* $f \in A^p_\alpha$ *if and only if*

$$\int_{\mathbb{B}_n} |f(z)|^{p-q} \left[(1-|z|^2) \left| \frac{\partial f}{\partial z_k}(z) \right| \right]^q dv_\alpha(z) < \infty \tag{9}$$

for all $1 \le k \le n$.

Proof. It is clear from the definition of $|\nabla f(z)|$ that for a holomorphic function f in $\mathbb{B}_n$, condition (c) in Theorem 2 is equivalent to the condition in (9). □

Finally we use an example to show that the range $0 < q < p + 2$ in Theorem 2 is best possible. Simply take $f(z) = z_1$. Then on the compact set $|z| \le 1/2$, we have $|\widetilde{\nabla} f(z)| \sim |\nabla f(z)| = 1$. It follows that

$$\begin{aligned}\int_{|z|<1/2} |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q \, dv_\alpha(z) &\sim \int_{|z|<1/2} |f(z)|^{p-q} \, dv_\alpha(z) \\ &= \int_{|z|<1/2} |z_1|^{p-q} \, dv_\alpha(z).\end{aligned}$$

By integration in polar coordinates (see Lemma 1.8 of [3] for example), the last integral above is comparable to

$$\int_0^{1/2} r^{2n-1+p-q} \, dr \int_{\mathbb{S}_n} |\zeta_1|^{p-q} \, d\sigma(\zeta).$$

If $q \ge p + 2$, the product above is always infinite. In fact, if $n = 1$, then

$$\int_0^{1/2} r^{2n-1+p-q} \, dr = \infty;$$

if $n \ge 2$, then by a well-known formula for evaluating integrals of functions of fewer variables on the unit sphere (see Lemma 1.9 of [3] for example), we have

$$\int_{\mathbb{S}_n} |\zeta_1|^{p-q} \, d\sigma(\zeta) = c \int_{\mathbb{D}} |w|^{p-q} (1 - |w|^2)^{n-2} \, dA(w) = \infty,$$

where c is a positive constant and dA is area measure on the unit disk $\mathbb{D}$. This shows that the range $q < p + 2$ is best possible in Theorem 2 as well as in Lemma 10.

REFERENCES

[1] C. Ouyang, W. Yang, and R. Zhao, Characterizations of Bergman spaces and Bloch space in the unit ball of $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **347** (1995), 4301-4313.

[2] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[3] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2004.

MIROSLAV PAVLOVIĆ, MATEMATIČKI FAKULTET, STUDENTSKI TRG 16, 11001 BELGRADE, P.P. 550, SERBIA

E-mail address: pavlovic@matf.bg.ac.yu

KEHE ZHU, DEPARTMENT OF MATHEMATICS, SUNY, ALBANY, NY 12222, USA

E-mail address: kzhu@math.albany.edu