

# DERIVATIVE-FREE CHARACTERIZATIONS OF $Q_K$ SPACES

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ABSTRACT. We give two characterizations of the Möbius invariant  $Q_K$  spaces, one in terms of a double integral and the other in terms of the mean oscillation in the Bergman metric. Both characterizations avoid the use of derivatives. Our results are new even in the case of  $Q_p$  spaces.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ . For any  $0 \leq p < \infty$  we consider the space  $Q_p$  consisting of all analytic functions  $f$  in  $\mathbb{D}$  such that

$$\|f\|_{Q_p}^2 = \sup_{\varphi} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi(z)|^2)^p dA(z) < \infty,$$

where  $dA$  is area measure on  $\mathbb{D}$  normalized so that  $A(\mathbb{D}) = 1$ , and the supremum is taken over  $\varphi \in \text{Aut}(\mathbb{D})$ , the group of Möbius maps of  $\mathbb{D}$ . The space  $Q_p$  is Möbius invariant in the sense that  $\|f \circ \varphi\|_{Q_p} = \|f\|_{Q_p}$  for every  $f \in Q_p$  and  $\varphi \in \text{Aut}(\mathbb{D})$ .

Since every Möbius map  $\varphi$  can be written as

$$\varphi(z) = e^{i\theta} \varphi_a(z),$$

where  $\theta$  is real and

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

is the Möbius map of the unit disk that interchanges the points 0 and  $a$ , we can also write

$$\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dA(z).$$

It is well known that for  $p > 1$ , we have  $Q_p = \mathcal{B}$ , the Bloch space of analytic functions  $f$  in  $\mathbb{D}$  such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

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It is also well known that  $Q_p = \text{BMOA}$  when  $p = 1$ . When  $p = 0$ , the space  $Q_p$  degenerates to the Dirichlet space. See [9] for a summary of recent research about  $Q_p$  spaces.

More generally, for any nonnegative and Lebesgue measurable function  $K$  on  $(0, 1]$ , we consider the space  $Q_K$  consisting of all analytic functions  $f$  in the unit disk such that

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) < \infty.$$

Clearly, if  $K(t) = t^p$ , then  $Q_K = Q_p$ . It is also clear that  $Q_K$  is Möbius invariant, that is,  $\|f \circ \varphi\|_{Q_K} = \|f\|_{Q_K}$  whenever  $f \in Q_K$  and  $\varphi \in \text{Aut}(\mathbb{D})$ . A great deal of function theory is worked out in [3] for the spaces  $Q_K$ . The paper [8] characterizes  $Q_K$  spaces in terms of higher order derivatives. More research on  $Q_K$  spaces can be found in [2][7].

The purpose of this paper is to give two characterizations of  $Q_K$  that is free of the use of derivatives. The first characterization, Theorem 6, is in terms of a double integral involving the area measure on  $\mathbb{D}$ . The other characterization, Theorem 8, is based on the mean oscillation of a function in the Bergman metric. Our results are new even for  $Q_p$  spaces, although the corresponding results for the Bloch space are well known. See [1] and [10].

As a consequence of our main results, the Bloch space and the space BMOA are characterized by the same type of conditions involving the area measure. For example, BMOA can now be characterized in terms of the mean oscillation in the Bergman metric. This seems to be something that has not been noticed before.

## 2. PRELIMINARIES

We need an elementary but somewhat non-standard tool, the so-called oro-coordinates on the unit disk. Recall that for any  $r > 0$  the equation

$$\frac{1 - |z|^2}{|1 - z|^2} = r$$

defines a circle  $C_r$  that is internally tangent to the unit circle  $|z| = 1$  at the point  $z = 1$ . These are called oro-cycles at  $z = 1$ ; see [4]. It is easy to check that  $C_r$  can be rewritten as

$$\left| z - \frac{r}{1+r} \right| = \frac{1}{1+r}.$$

It follows that the unit disk  $\mathbb{D}$  can be parametrized by

$$z = \frac{r}{1+r} + \frac{1}{1+r} e^{i\theta}, \quad 0 \leq r < \infty, 0 \leq \theta \leq 2\pi.$$

This reparametrization of the unit disk will be called oro-coordinates on  $\mathbb{D}$ . The following lemma gives the right form of the area integral in terms of oro-coordinates.

**Lemma 1.** *Suppose  $f$  is Lebesgue measurable on  $\mathbb{D}$ . If  $f$  is nonnegative or belongs to  $L^1(\mathbb{D}, dA)$ , then*

$$\int_{\mathbb{D}} f(z) dA(z) = \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} f(r, \theta) \frac{1 - \cos \theta}{(1+r)^3} dr d\theta,$$

where  $f(r, \theta)$  is the function  $f$  in oro-coordinates.

*Proof.* See [6]. □

If the function  $K$  is only defined on  $(0, 1]$ , then we extend it to  $(0, \infty)$  by setting  $K(t) = K(1)$  for  $t > 1$ . We can then define an auxiliary function as follows:

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

We further assume that  $K$  is continuous and nondecreasing on  $(0, 1]$ . This ensures that the function  $\varphi_K$  is continuous and nondecreasing on  $(0, \infty)$ .

We will also need to use the so-called Berezin transform. More specifically, for any function  $f \in L^1(\mathbb{D}, dA)$ , we define a function  $Bf$  by

$$Bf(z) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} f(w) dA(w), \quad z \in \mathbb{D}.$$

It is standard terminology nowadays to call  $Bf$  the Berezin transform of  $f$ . By a change of variables, we can also write

$$Bf(z) = \int_{\mathbb{D}} f \circ \varphi_z(w) dA(w), \quad z \in \mathbb{D}.$$

See [5] and [10] for basic properties of the Berezin transform.

The following estimate is the key to the main results of the paper.

**Lemma 2.** *Let  $K$  be any nonnegative and Lebesgue measurable function on  $(0, \infty)$  and  $f(z) = K(1 - |z|^2)$ . If*

$$\int_0^\infty \frac{\varphi_K(x)}{(1+x)^3} dx < \infty, \tag{1}$$

then there exists a positive constant  $C$  such that  $Bf(z) \leq Cf(z)$  for all  $z \in \mathbb{D}$ .

*Proof.* Recall that

$$Bf(z) = \int_{\mathbb{D}} K(1 - |\varphi_z(w)|^2) dA(w).$$

Since

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\bar{w}|^2},$$

it follows from the definition of  $\varphi_K$  that

$$K(1 - |\varphi_z(w)|^2) \leq K(1 - |z|^2)\varphi_K\left(\frac{1 - |w|^2}{|1 - z\bar{w}|^2}\right),$$

so

$$Bf(z) \leq f(z)\Phi(z), \quad z \in \mathbb{D},$$

where

$$\Phi(z) = \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - z\bar{w}|^2}\right) dA(w).$$

The function  $\Phi$  is continuous on  $\mathbb{D}$ , so it is bounded on any compact subset of  $\mathbb{D}$ . On the other hand, if  $z \in \mathbb{D}$  is nonzero, then we use the monotonicity of  $\varphi_K$  (together with the obvious inequality  $1 - |w|^2 \leq 1 - |zw|^2$ ) and a change of variables to obtain

$$\begin{aligned} \Phi(z) &\leq \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |z\bar{w}|^2}{|1 - z\bar{w}|^2}\right) dA(w) \\ &= \frac{1}{|z|^2} \int_{|w| < |z|} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w) \\ &\leq \frac{1}{|z|^2} \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w). \end{aligned}$$

If the condition in (1) is satisfied, then an application of Lemma 1 shows that

$$\int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w) = 2 \int_0^\infty \frac{\varphi_K(r)}{(1+r)^3} dr < \infty.$$

Combining this with the estimates in the previous paragraphs, we conclude that the function  $\Phi(z)$  is bounded in  $\mathbb{D}$ . This completes the proof of the lemma.  $\square$

If  $K(t) = t^p$  and  $f(z) = K(1 - |z|^2)$ , then the Berezin transform of  $f$  is

$$\begin{aligned} Bf(z) &= \int_{\mathbb{D}} (1 - |\varphi_z(w)|^2)^p dA(w) \\ &= (1 - |z|^2)^p \int_{\mathbb{D}} \frac{(1 - |w|^2)^p dA(w)}{|1 - z\bar{w}|^{2p}} \\ &= f(z) \int_{\mathbb{D}} \frac{(1 - |w|^2)^p dA(w)}{|1 - z\bar{w}|^{2p}}. \end{aligned}$$

By a well-known estimate (see Lemma 4.2.2 of [10] for example), the inner integral above is a bounded function of  $z$  if and only if  $p < 2$ .

On the other hand, if  $K(t) = t^p$ , then it is easy to see that  $\varphi_K(t) = t^p$  as well. In this case, it is clear that condition (1) holds if and only if  $p < 2$ . Therefore, at least in the case of  $Q_p$  spaces, the condition in (1) is best possible.

Let  $\beta(z, w)$  denote the Bergman metric between two points  $z$  and  $w$  in  $\mathbb{D}$ . It is well known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For  $z \in \mathbb{D}$  and  $R > 0$  we use

$$D(z, R) = \{w \in \mathbb{D} : \beta(z, w) < R\}$$

to denote the Bergman metric ball at  $z$  with radius  $R$ . If  $R$  is fixed, then it can be checked that the area of  $D(z, R)$ , denoted by  $|D(z, R)|$ , is comparable to  $(1 - |z|^2)^2$  as  $z$  approaches the unit circle. See [10].

**Lemma 3.** *For any  $R > 0$  there exists a positive constant  $C$  (depending on  $R$ ) such that*

$$|f(z)|^2 \leq \frac{C}{|D(z, R)|} \int_{D(z, R)} |f(w)|^2 dA(w)$$

for all  $z \in \mathbb{D}$  and all analytic functions  $f$  in  $\mathbb{D}$ .

*Proof.* See Lemma 4.3.8 of [10] for example. □

### 3. A DOUBLE INTEGRAL CHARACTERIZATION OF $Q_K$

In this section we characterize the spaces  $Q_K$  in terms of a double integral that does not involve the use of derivatives. We begin with the following estimate of Dirichlet type integrals.

**Lemma 4.** *There exists a constant  $C > 0$  (independent of  $K$ ) such that*

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) \leq CI(f)$$

for all analytic functions  $f$  in  $\mathbb{D}$ , where

$$I(f) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |z|^2) dA(z) dA(w).$$

*Proof.* We write the double integral  $I(f)$  as an iterated integral

$$I(f) = \int_{\mathbb{D}} \frac{K(1 - |z|^2)}{(1 - |z|^2)^2} dA(z) \int_{\mathbb{D}} \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} |f(z) - f(w)|^2 dA(w).$$

Making a change of variables in the inner integral, we obtain

$$I(f) = \int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^2} dA(z) \int_{\mathbb{D}} |f(\varphi_z(w)) - f(z)|^2 dA(w). \quad (2)$$

It is well known (verify using Taylor expansion or see Theorem 4.27 of [10], for example) that

$$\int_{\mathbb{D}} |g(w) - g(0)|^2 dA(w) \sim \int_{\mathbb{D}} |g'(w)|^2 (1-|w|^2)^2 dA(w) \quad (3)$$

for analytic functions  $g$  in  $\mathbb{D}$ . Applying (3) to the inner integral in (2) with the function  $g(w) = f(\varphi_z(w))$ , we see that

$$I(f) \sim \int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^2} dA(z) \int_{\mathbb{D}} |(f \circ \varphi_z)'(w)|^2 (1-|w|^2)^2 dA(w).$$

By the chain rule and a change of variables, we get

$$I(f) \sim \int_{\mathbb{D}} K(1-|z|^2) dA(z) \int_{\mathbb{D}} \frac{|f'(w)|^2 (1-|w|^2)^2}{|1-z\bar{w}|^4} dA(w). \quad (4)$$

Fix any positive radius  $R$ . Then there exists a constant  $C > 0$  such that

$$I(f) \geq C \int_{\mathbb{D}} K(1-|z|^2) dA(z) \int_{D(z,R)} \frac{|f'(w)|^2 (1-|w|^2)^2}{|1-z\bar{w}|^4} dA(w).$$

It is well known that

$$\frac{(1-|w|^2)^2}{|1-z\bar{w}|^4} \sim \frac{1}{(1-|z|^2)^2} \sim \frac{1}{|D(z,R)|}$$

for  $w \in D(z,R)$ ; see 4.3 of [10]. It follows that there exists a positive constant  $C$  such that

$$I(f) \geq C \int_{\mathbb{D}} K(1-|z|^2) dA(z) \frac{1}{|D(z,R)|} \int_{D(z,R)} |f'(w)|^2 dA(w).$$

Combining this with Lemma 3, we obtain a positive constant  $C$  such that

$$I(f) \geq C \int_{\mathbb{D}} |f'(z)|^2 K(1-|z|^2) dA(z).$$

This completes the proof of the lemma.  $\square$

The second half of the above proof (the one based on Lemma 3) also follows from inequality (5.5) of [3]; see the proof of Lemma 5 below. But the proof of (5.5) in [3] transfers the setting to the upper half-plane and makes use of certain additional assumptions on  $K$ , while our proof here does not depend on any special property of  $K$ . The following lemma, however, makes a key assumption on  $K$ .

**Lemma 5.** *If the function  $K$  satisfies condition (1), then there exists a constant  $C > 0$  such that*

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) \geq CI(f)$$

for all analytic functions  $f$  in  $\mathbb{D}$ , where  $I(f)$  is as given in Lemma 4.

*Proof.* By Fubini's theorem, we can rewrite (4) as

$$I(f) \sim \int_{\mathbb{D}} |f'(w)|^2 dA(w) \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - z\bar{w}|^4} K(1 - |z|^2) dA(z).$$

The inner integral above is nothing but the Berezin transform of the function  $K(1 - |z|^2)$  at the point  $w$ . The desired estimate now follows from Lemma 2.  $\square$

We can now prove the main result of this section.

**Theorem 6.** *Suppose  $K$  satisfies the condition in (1). Then an analytic function  $f$  in  $\mathbb{D}$  belongs to  $Q_K$  if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) < \infty. \quad (5)$$

*Proof.* Recall that  $f \in Q_K$  if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) < \infty.$$

By a change of variables, we have  $f \in Q_K$  if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 K(1 - |z|^2) dA(z) < \infty.$$

Replacing  $f$  by  $f \circ \varphi_a$  in Lemmas 4 and 5, we conclude that  $f \in Q_K$  if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^2}{|1 - z\bar{w}|^4} K(1 - |z|^2) dA(z) dA(w) < \infty.$$

Changing variables and simplifying the result, we find that the double integral above is the same as

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w).$$

Therefore,  $f \in Q_K$  if and only if the condition in (5) holds.  $\square$

## 4. THE MEAN OSCILLATION IN THE BERGMAN METRIC

In this section we give two closely related characterizations of  $Q_K$  spaces, one in terms of the Berezin transform and the other in terms of the mean oscillation in the Bergman metric.

Given a function  $f \in L^2(\mathbb{D}, dA)$  it is customary to write

$$MO(f)(z) = \sqrt{B(|f|^2)(z) - |Bf(z)|^2};$$

see Section 7.1 in [10]. It is easy to check that

$$\begin{aligned} [MO(f)(z)]^2 &= \int_{\mathbb{D}} |f \circ \varphi_z(w) - Bf(z)|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w) - Bf(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w). \end{aligned}$$

If the function  $f$  is analytic, then it is easy to see that  $Bf = f$ , so that

$$\begin{aligned} [MO(f)(z)]^2 &= \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w) - f(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w). \end{aligned}$$

We can think of  $MO(f)(z)$  as the invariant mean oscillation of  $f$  in the Bergman metric at the point  $z$ , because we always have

$$MO(f \circ \varphi)(z) = MO(f)(\varphi(z)),$$

where  $\varphi \in \text{Aut}(\mathbb{D})$ .

We can now reformulate Theorem 6 as follows, which is in the same spirit as condition (i) in Theorem 6.1 of [3].

**Theorem 7.** *If  $K$  satisfies condition (1), then an analytic function  $f$  in  $\mathbb{D}$  belongs to  $Q_K$  if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty, \quad (6)$$

where

$$d\tau(z) = \frac{dA(z)}{(1 - |z|^2)^2}$$

is the Möbius invariant measure on the unit disk.

*Proof.* Consider the following integrals from Theorem 6

$$I_a(f) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w).$$

We rewrite it as an iterated integral

$$I_a(f) = \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) dA(z) \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} dA(w),$$

or

$$I_a(f) = \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) d\tau(z) \int_{\mathbb{D}} |f(w) - f(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w).$$

According to the calculations preceding this theorem, we have

$$I_a(f) = \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z).$$

This proves the desired result.  $\square$

Fix a positive radius  $R$  and denote by

$$A_R(f)(z) = \frac{1}{|D(z, R)|} \int_{D(z, R)} f(w) dA(w)$$

the average of  $f$  over the Bergman metric ball  $D(z, R)$ . We define

$$MO_R(f)(z) = \left[ \frac{1}{|D(z, R)|} \int_{D(z, R)} |f(w) - A_R(f)(z)|^2 dA(w) \right]^{\frac{1}{2}}$$

and call it the mean oscillation of  $f$  in the Bergman metric at the point  $z$ . It is easy to verify that

$$[MO_R(f)(z)]^2 = A_R(|f|^2)(z) - |A_R(f)(z)|^2.$$

See Section 7.1 of [10] for basic facts about the mean oscillation in the Bergman metric.

The Möbius invariant mean oscillation  $MO(f)$  and the localized mean oscillation  $MO_R(f)$  in the Bergman metric are useful in the study of Hankel operators on the Bergman space. See [10] and [11].

**Theorem 8.** *If  $K$  satisfies condition (1), then an analytic function  $f$  in  $\mathbb{D}$  belongs to  $Q_K$  if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MO_R(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty, \quad (7)$$

where  $R$  is any fixed positive radius.

*Proof.* There exists a positive constant  $C$  (depending on  $R$  only) such that

$$MO_R(f)(z) \leq MO(f)(z), \quad z \in \mathbb{D},$$

where  $f$  is any function in  $L^2(\mathbb{D}, dA)$ . See the proof of Theorem 7.1.6 in [10]. Therefore, condition (6) implies condition (7).

On the other hand, since  $D(0, R)$  is an Euclidean disk centered at the origin, we can find a positive constant  $C$  (depending on  $R$  only) such that

$$|f'(0)|^2 \leq C \int_{D(0,R)} |f(w) - c|^2 dA(w)$$

for all analytic  $f$  in  $\mathbb{D}$  and all complex constants  $c$ . Replace  $f$  by  $f \circ \varphi_z$  and replace  $c$  by  $A_R(f)(z)$ . Then

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C \int_{D(0,R)} |f \circ \varphi_z(w) - A_R(f)(z)|^2 dA(w).$$

Make an obvious change of variables on the right hand side. Then

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C \int_{D(z,R)} |f(w) - A_R(f)(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} dA(w).$$

Since

$$\frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} \sim \frac{1}{(1 - |z|^2)^2} \sim \frac{1}{|D(z, R)|}$$

for  $w \in D(z, R)$ , we can find another positive constant  $C$  such that

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C [MO_R(f)(z)]^2, \quad z \in \mathbb{D}.$$

It follows that for each  $a \in \mathbb{D}$  the integral

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z)$$

is less than or equal to  $C$  times the integral

$$\int_{\mathbb{D}} [MO_R(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z).$$

This shows that the condition in (7) implies  $f \in Q_K$ .  $\square$

Recall from [3] that a positive Borel measure  $\mu$  on  $\mathbb{D}$  is called a  $K$ -Carleson measure if

$$\sup_I \int_{S(I)} K \left( \frac{1 - |z|}{|I|} \right) d\mu(z) < \infty,$$

where the supremum is taken over all sub-arcs  $I \subset \partial D$ . Here, for a sub-arc  $I$  of  $\partial \mathbb{D}$ ,  $|I|$  is the length of  $I$  and

$$S(I) = \{r\zeta : \zeta \in I, 1 - |I| < r < 1\}$$

is the corresponding Carleson “square”.

**Theorem 9.** *Suppose  $K$  satisfies the following two conditions:*

- (a) *There exists a constant  $C > 0$  such that  $K(2t) \leq CK(t)$  for all  $t > 0$ .*

(b) *The auxiliary function  $\varphi_K$  has the property that*

$$\int_0^1 \varphi_K(s) \frac{ds}{s} < \infty.$$

*Let  $\mu$  be a positive Borel measure on  $\mathbb{D}$ . Then  $\mu$  is a  $K$ -Carleson measure if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) d\mu(z) < \infty.$$

*Proof.* See Corollary 3.2 of [3]. □

Since  $Q_K$  is defined by the condition

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) < \infty,$$

we see that  $f \in Q_K$  if and only if the measure  $|f'(z)|^2 dA(z)$  is a  $K$ -Carleson measure. This is one of the conclusions of Theorem 3.1 in [3]. The following corollary gives two analogous characterizations.

**Corollary 10.** *Suppose  $K$  satisfies condition (1) and conditions (a) and (b) in Theorem 9. Let  $R > 0$  be a fixed radius. Then the following conditions are equivalent for an analytic function  $f$  in  $\mathbb{D}$ .*

- (a) *The function  $f$  belongs to  $Q_K$ .*
- (b) *The measure*

$$d\mu(z) = [MO(f)(z)]^2 d\tau(z)$$

*is a  $K$ -Carleson measure.*

- (c) *The measure*

$$d\nu(z) = [MO_R(f)(z)]^2 d\tau(z)$$

*is a  $K$ -Carleson measure.*

*Proof.* This is a direct consequence of Theorems 7, 8, and 9. □

## 5. THE SPACES $Q_{K,0}$

The space  $Q_{K,0}$  consists of analytic functions  $f$  in  $\mathbb{D}$  with the property that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) = 0.$$

It can be checked that  $Q_{K,0}$  is a closed subspace in  $Q_K$ .

**Theorem 11.** *If  $K$  satisfies condition (1) and  $R > 0$  is fixed, then the following conditions are equivalent for all analytic functions  $f$  in  $\mathbb{D}$ .*

- (1)  $f \in Q_{K,0}$ .

(2)

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) = 0.$$

(3)

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0.$$

(4)

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{D}} [MO_R(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0.$$

*Proof.* All our earlier estimates are pointwise estimates with respect to  $a \in \mathbb{D}$ . So the little oh version of these results must be valid.  $\square$

A positive Borel measure  $\mu$  on  $\mathbb{D}$  is called a vanishing  $K$ -Carleson measure if

$$\lim_{|I| \rightarrow 0} \int_{S(I)} K \left( \frac{1 - |z|}{|I|} \right) d\mu(z) = 0.$$

Carefully checking the proof of Theorem 3.1 in [3], we see that the little oh version of Theorem 9 holds as well, from which we obtain the following.

**Theorem 12.** *Under the same assumptions in Theorem 10, the following conditions are equivalent for analytic functions  $f$  in  $\mathbb{D}$ .*

- (a) *The function  $f$  belongs to  $Q_{K,0}$ .*
- (b) *The measure*

$$d\mu(z) = |f'(z)|^2 dA(z)$$

*is a vanishing  $K$ -Carleson measure.*

- (c) *The measure*

$$d\mu(z) = [MO(f)(z)]^2 d\tau(z)$$

*is a vanishing  $K$ -Carleson measure.*

- (d) *The measure*

$$d\mu(z) = [MO_R(f)(z)]^2 d\tau(z)$$

*is a vanishing  $K$ -Carleson measure.*

*Proof.* We leave the details to the interested reader.  $\square$

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