Selected Homework Exercise Solutions  
Math 331, Transformation Geometry  

February 6, 2002

P. 17, no. 6: Prove that if $S$ is on segment $PR$ and $T$ is on segment $QR$, the segments $PT$ and $QS$ intersect.

Response. The exercise is mainly meaningful when $S$ and $T$ are not endpoints of the segments on which they lie and when $P, Q, R$ are not collinear. If that is the case, then there are numbers $s, t$ with $0 < s, t < 1$ such that $S = (1 - s)R + sP$ and $T = (1 - t)R + tQ$. Moreover, a point on the line $PT$ has the form $(1 - x)P + xT$ for some $x$, while a point on the line $QS$ has the form $(1 - y)Q + yS$ for some $y$. The lines $PT$ and $QS$ meet if and only if there are numbers $x, y$ for which the two previous expressions are equal. The question of whether such values of $x, y$ exist (and, hence, the lines intersect) is addressed algebraically.

If those expressions are expanded using the formulas for $S$ and $T$, then their equality becomes the relation

$$(1 - x)P + xtQ + x(1 - t)R = ysP + (1 - y)Q + y(1 - s)R.$$ 

With the assumption that $P, Q, R$ are not collinear, hence, barycentrically independent, the corresponding coefficients of $P, Q, R$ in this relation must be equal. Hence,

$$1 - x = sy, \quad tx = 1 - y, \quad (1 - t)x = (1 - s)y.$$

Solving these equations simultaneously for $x, y$ one finds

$$x = \frac{1 - s}{1 - st}, \quad y = \frac{1 - t}{1 - st}.$$

The fact that these solutions exist means that the lines $PT$ and $QS$ intersect. Moreover, from the fact that $0 < s, t < 1$ it is clear that $0 < x, y < 1$, and, therefore, that the point where the lines intersect is the intersection of the segments $PT$ and $QS$.

P. 31, no. 4: $P$ is a point inside a given triangle $ABC$, and $F$ is the point on the side $AB$ where the line $CP$ meets $AB$. $D$ is the point of intersection with $AC$ of the line through $P$ parallel to $BC$, and $E$ is the point of intersection with $BC$ of the line through $P$ parallel to $AC$. Prove that $|AF| \cdot |CD| \cdot |BC| = |BF| \cdot |CE| \cdot |AC|$.

Response. If the vertices $A, B, C$ are arranged clockwise, then each of the sides of the triangle is divided into two segments by the points $F, E, D$. Each corresponding length $a, b, c$ is then decomposed:

$a = a' + a'', \quad b = b' + b'', \quad c = c' + c''$, where $a' = |CE|, \quad b' = |AD|$, and $c' = |BF|$. With this notation the task is to show that $ab''c'' = bc'a'$.

Let $P = uA + vB + wC$. Since the point $F$ has unique barycentric coordinates with respect to $A, B, C$ and is both a barycentric combination of the two points $C, P$ and also a barycentric combination of the two points $A, B$, one sees that

$$F = \frac{u}{u + v}A + \frac{v}{u + v}B.$$

Let $D = (1 - s)C + sA$ and $E = (1 - t)C + tB$. By the parallelogram law of addition

$$P = D + E - C,$$

which leads to a second barycentric expression for $P$ relative to the three vertices:

$$P = sA + tB + (1 - s - t)C.$$

Hence, $s = u, \quad t = v$, and, therefore,

$$a' = va, \quad a'' = (1 - v)a, \quad b' = (1 - u)b, \quad b'' = ub,$$

while

$$c' = \frac{u}{u + v}c, \quad c'' = \frac{v}{u + v}c.$$

Thus,

$$ab''c'' = \frac{uv}{u + v}abc = bc'a'.$$