**Topology Prelim**

**Sept 3, 1999**

**Do as many problems as possible.**

1. Suppose that $X$ is metrizable. Prove that $X$ is $2^{\text{nd}}$ countable if and only if $X$ contains a countable dense subset.

2. Suppose that $X$ is a topological space, $U$ is open in $X$ and $A$ is dense in $X$. Prove that $U \subset cl(A \cap U)$. Here $cl$ stands for closure.

3. Suppose that $X$ is a compact metric space with metric $d$. Suppose that $f : X \to X$ is a function so that $d(x, y) = d(f(x), f(y))$ for all $x, y$ in $X$. Prove that $f$ is onto.

4. a. Suppose that $X$ has a finite number of components. Prove that each component is open.

   b. Give an example to show that the conclusion is false if $X$ has an infinite number of components.

5. Let $J$ be an index set and for each $j \in J$ suppose that $X_j$ is homeomorphic to [0,1]. Under what conditions on $J$ is $\Pi X_j$ metrizable. Prove your answer.

6. Let $R$ be given the half open interval topology where the basis consists of intervals closed on the left and open on the right. Which of the following subset are compact.

   a. $[0, 1]$

   b. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0\}$

7. Suppose that $X$ and $Y$ are both homeomorphic to $S^2$. Let $Z$ be the space obtained when the north pole of $X$ is identified to the south pole of $Y$ and the north pole of $Y$ is identified to the the south pole of $X$.

   a. What is $\Pi_1(Z)$?

   b. Describe the universal cover of $Z$.

8. Suppose that $X$ is Hausdorff and $Y$ is compact. Let $f : X \to Y$ be a continuous function which is 1-1 and onto. Must $f$ be a homeomorphism?