Topology Prelim

June 3, 1998

Do as many problems as possible.

1. Let X be a set. Prove there exists a smallest topology on X which makes X a T_1 space (points are closed sets).

2. Let X be a topological space and let $Y_1, Y_2, \ldots$ be a sequence of connected subspaces so that

   1. $X = \bigcup_{i=1}^{\infty} Y_i$,

   2. $Y_i \cap Y_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots$.

   Prove that X is connected.

3. a. Prove that if $f : [-1, 1] \to [-1, 1]$ is continuous, $f(-1) = -1$ and $f(1) = 1$ then f is onto.

   b. Prove that if $f : D^2 \to D^2$ is continuous and $f|S^1$ is the identity then f is onto.

4. Let A be any finite non-empty subset of R^2.

   a. Prove that $R^2 - A$ is connected.

   b. Prove that $R^2 - A$ is not simply connected.

5. a. Prove or disprove: If A is a connected subset of $\mathbf{R}^2$, then $\overline{A}$ is connected.

   b. Prove or disprove: If A is a path connected subset of $\mathbf{R}^2$, then $\overline{A}$ is path connected.

6. Let $X = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \cup \{(0, 0, z) : -1 \leq z \leq 1\}$.

   a. What is the universal covering space of X?

   b. Compute $\pi_1(X, (1, 0, 0))$.

7. Let X be a compact Hausdorff space and let $C[X] = \{f : f : X \to R^1 \text{ is continuous}\}$.

   Let $d(f, g) = \sup\{|f(x) - g(x)|, x \in X\}$. Prove that d is a metric on $C[X]$.

8. Let $X = \prod_{i=1}^{\infty} \{0, 1\}$ with the product topology where $\{0, 1\}$ is given the discrete topology.

   Let $A = \{x_1, x_2, \ldots | x_j = 1 \text{ for only a finite number of } j\}$. Prove or disprove that A is closed.