

# TOPOLOGY PRELIMINARY EXAMINATION

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Do as many problems as possible.

1. An isometry of a metric space  $X$  is a map  $h : X \rightarrow X$  so that  $d(h(x), h(y)) = d(x, y)$  for all  $x, y \in X$ . Prove that an isometry of a compact metric space is necessarily surjective.

2. Let  $C_n$  be the subspace of  $\mathbf{R}^2$  defined by  $C_n = \{(x, y) : (x - \frac{1}{n})^2 + y^2 = (\frac{1}{n})^2\}$ . Let  $Y$  be the subspace

$$Y = \bigcup_{n \in \mathbf{Z}_+} C_n$$

of  $\mathbf{R}^2$ , and let  $X$  be the subspace  $C_1 \times \mathbf{Z}_+$  of  $\mathbf{R}^2 \times \mathbf{R}$ . Define  $g : X \rightarrow Y$  by  $g((x, y), n) = (\frac{x}{n}, \frac{y}{n})$ . Show  $g$  is continuous and surjective, but  $g$  is not a quotient map.

3. Prove that  $f : X \rightarrow Y$  is continuous if and only if for all  $A \subset Y$

$$\overline{f^{-1}(A)} \subset f^{-1}(\overline{A}) .$$

4. Prove:

- (a) The union of a collection of connected sets with a point in common is connected.
- (b)  $X, Y$  connected  $\iff X \times Y$  connected.

5. Show that a compact, Hausdorff,  $2^{nd}$  countable space may be embedded in the countable product of copies of  $[0, 1]$ .

6. Show that the rationals  $\mathbf{Q}$  are not locally compact.

7. Let  $p : X \rightarrow Y$  be a covering, with  $Y$  connected. Prove that if  $\text{card}\{p^{-1}(y)\} = k \in \mathbf{Z}_+$  for some point  $y \in Y$ , then  $\text{card}\{p^{-1}(y)\} = k$  for all points  $y \in Y$ .

8. Let  $X = \mathbf{R}P^2 \vee \mathbf{R}P^2$ , the one point union of two real projective planes.

- (a) Find  $\pi_1(X)$ .
- (b) Describe the universal cover of  $X$  and the covering transformations.
- (c) Does  $\pi_1(X)$  have elements of infinite order? Explain.