

Ph.D. Preliminary Exam in Topology

August 26, 1994

Do all problems.

1. Prove that the set \mathbf{Q} of rational numbers cannot be the set of zeros of any continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$. Can the Cantor set be the set of zeros of such a function? Why?

2. Are any of the following spaces homeomorphic? Justify your answer.

(a) $S^1 \times S^1 \times S^1 \times \dots$ (countable product of S^1 , product topology)

(b)
$$= \bigcup_{n=1}^{\infty} C_n$$
 where C_n is the circle of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$ (The Hawaiian earring) (relative topology from \mathbf{R}^2)

(c) $\bigvee_{n=1}^{\infty} S^1$ The disjoint union of countably many copies of S^1 with basepoints

identified. (identification \equiv quotient topology)

(d) The infinite cheese slicer with vertical lines at $x = \frac{1}{n}$ for $n = 1, 2, 3, \dots$, and at $x = 0$. (relative topology from \mathbf{R}^2)

3. Let X be the quotient space of the 2-simplex

at right obtained by identifying the edges as shown.

Is X simply connected? Is it homeomorphic to S^2 ? Justify.

4. Find examples of each of the following. For one of them, show why your example has/lacks the indicated properties.
- (a) A compact Hausdorff space which is not metrizable.
 - (b) A connected, non-path-connected space.
 - (c) A non-Hausdorff space X in which each point has a neighborhood homeomorphic to the interval $(0,1)$.
5. Let $\{X_\alpha \mid \alpha \in A\}$ be a family of topological spaces, and $X = \prod_{\alpha \in A} X_\alpha$ the product set.
- (a) Define the product and box topologies on X . Suppose $A = \mathbf{N}$ and $X_\alpha = [0, 1]$ for each α , so that $X = [0, 1] \times [0, 1] \times \dots$
 - (b) Is X compact in the product topology? In the box topology?
 - (c) Is X connected in the product topology? In the box topology?

Verify your answer for either (b) or (c).

6. (a) Using covering spaces, prove that the fundamental group of the figure eight is not abelian.
- (b) What is the universal cover of $T^2 \times \mathbf{R}P^2$, where T^2 is the torus and $\mathbf{R}P^2$ is the projective plane?
 - (c) Describe the universal cover of the figure eight.