# Topology Prelim

## August 24, 2001

1. Prove that $X \times Y$ is connected if and only if both $X$ and $Y$ are connected.

2. a. Assume that $Y$ is compact. Prove that $p_x : X \times Y \to X$ is closed ($p_x$ is the projection onto $X$).

   b. Show that the assumption that $Y$ is compact is necessary.

3. Let $X$ be Hausdorff and $Y$ be compact Hausdorff. Then $f : X \to Y$ is continuous if and only if the graph of $f$ is closed.

4. For each $\alpha$ in $A$ let $X\alpha$ be a topological space. Prove that $\prod X_\alpha$ is discrete if and only if each $X_\alpha$ is discrete and equal to one point for all but a finite number of $\alpha$.

5. Prove that a compact metric space is second countable.

6. a. Let $f : X \to Y$ be a closed continuous surjection. Prove that if $U$ is an open set containing $f^{-1}(y)$ for some $y$ in $Y$ then there exists an open set $V$ containing $y$ such that

$$f^{-1}(V) \subset U$$

   b. Let $f : X \to Y$ be a closed continuous surjection such that $Y$ and $f^{-1}(y)$ are compact for all $y$ in $Y$. Prove that $X$ is compact.

7. Find $\prod_1(X)$ and the universal cover of $X$ where $X$ is the union of the subsets $A$, $B$ and $C$.

   $A = \{(x, y, z) | x^2 + y^2 \leq 1 \text{ and } z = 0 \text{ or } 1\}$

   $B = \{(-1/2, 0, z) | \text{ where } 0 \leq z \leq 1\}$

   $C = \{(1/2, 0, z) | \text{ where } 0 \leq z \leq 1\}$