University at Albany

Department of Mathematics and Statistics

Preliminary Exam

Real Analysis

August, 1999

Do as many problems as possible.

1. State the following:
   (a) The Monotone Convergence Theorem
   (b) Fatou's Lemma
   (c) Hölders Inequality
   (d) Fubini's Theorem
   (e) The Radon-Nikodym Theorem
   (f) The Riesz Representation Theorem

2. Prove either (a) or (b) from first principles.

3. (a) Let A be a subset of $[0,1]$. How is the (Lebesgue) outer measure m^*A defined?
   (b) What does it mean to say that a set E is (Lebesgue) measurable?
   (c) Prove that if $m^*E = 0$, then E is measurable.

4. (a) Define what it means for $f : [a,b] \to \mathbf{R}$ to be of bounded variation (B.V.).
   (b) Let $f : [-1,1] \to \mathbf{R}$ be given by $f(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$, $f(0) = 0$. Prove or disprove that f is of bounded variation.
   (c) State (without proof) two consequences of being B.V.

5. Suppose f is Lebesgue integrable on $[0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x) \sin n\ x\ dx = 0\ .$$

6. (a) What does it mean to say that a function $f : [0,1] \to \mathbf{R}$ is in $L^p([0,1])$?

   (b) Prove that if $1 < p < q$ and f is in $L^q([0,1])$, then also f is in $L^p([0,1])$.

   (c) Give an example of a function that is in $L^2([0,1])$ but not $L^3([0,1])$.

7. Let λ denote Lebesgue measure on $I = [0,1]$ and μ counting measure, both regarded as Borel measures on I.

   (a) Show that the diagonal $\Delta = \{(x,y)|x = y\}$ is measurable with respect to $\lambda \times \mu$ on the Borel sets of $I \times I$.

   (b) Let f be the characteristic function of Δ. Compute $\int_I \left(\int_I f d\lambda \right) d\mu$, $\int_I \left(\int_I f d\mu \right) d\lambda$, and $\int_{I \times I} f \; d\lambda \times d\mu$.

   (c) Reconcile part b with Fubini's Theorem.