The University at Albany

Department of Mathematics and Statistics

Ph.D. Program

Preliminary Examination in Real Analysis

Friday, June 11, 1999

PART I

Do problems 1 and 2:

1. State the following theorems:

   A. The Lebesgue Monotone Convergence Theorem,

   B. Fatou's Lemma,

   C. Egoroff's Theorem,

   D. The Fubini Theorem,

   E. Hölder's Inequality.

2. A. Give a definition of the Lebesgue integral.

   B. Sketch a proof of the Lebesgue Monotone Convergence Theorem assuming only the basic properties of measure theory.

PART II

Do 6 of the following 8 problems:

3. If f and g are Lebesgue measurable functions on $I = [0, 1]$ then $f(x) - g(x)$ is also a Lebesgue measurable function.

4. Let $\{f_n\}$ be a sequence of non-negative Lebesgue measurable function is on $I = [0, 1]$ with

$$f_1(x) \geq f_2(x) \geq f_3(x) \ldots \geq f_n(x) \geq \ldots \geq 0$$

for each $x \in I$ and assume $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Prove that for almost every $x \in I$,

$$\lim_{n\to\infty} f_n(x) = 0\ .$$

5. Does there exist a strictly increasing function defined on an interval I so that $f'(x) = 0$ almost everywhere on I? Prove your answer.

6. A. Let μ and ν be Borel measures on $[0,1]$. Define the Radon-Nikodym derivative of ν with respect to μ.

   B. Let λ and μ be Borel measures on $[0,1]$. Show that μ is absolutely continuous with respect to $\lambda + \mu$.

   C. Let λ and μ be Borel measures on $[0,1]$. Show that

$$\frac{d\lambda}{d(\lambda+\mu)} + \frac{d\mu}{d(\lambda+\mu)} = 1, \ (\lambda+\mu) \text{ almost everywhere } .$$

7. Suppose f is Lebesgue integrable on R^+ and let

$$g(x) = \int_0^\infty \frac{f(t)}{x+t}\, dt, \quad x > 0 .$$

   Is g continuous? Does g have a limit at $x \to \infty$? Is g differentiable? Prove your answers.

8. Let $\{f_n\}$ be a sequence of continuous functions on $I = [0,1]$. Prove that the set of points where $\lim_{n\to\infty} f_n(x) = 0$ is an $F_{\sigma\delta}$ Borel set.

9. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ such that for each n, $|E_n| \geq \delta > 0$. Suppose that c_n is a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} c_n \chi_{E_n}(x) < \infty$ for almost every $x \in [0,1]$. Show that $\sum_{n=1}^{\infty} c_n < \infty$.

10. If $\{f_n\}$ is a sequence of measurable functions on $I = [0,1]$ with $\int_0^1 |f_n(x)|^2 dx \leq 1$ for each f_n, and if $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in I$ prove that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)| dx = 0 .$$

   (Hint: Use Fatou's lemma and Egoroff's theorem.)