University at Albany

Department of Mathematics and Statistics

Preliminary Examination

Real Analysis

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Do as many problems as possible!

1. State the following theorems.

a. Monotone Convergence Theorem

b. Fatou's Lemma

c. Dominated Convergence Theorem

d. Hölder's Inequality

2. Use the Bounded Convergence Theorem to prove Fatou's Lemma and the Monotone Convergence Theorem for Lebesgue integration.

3. Prove that there is a subset of $[0,1]$ which is not Lebesgue measurable. Emphasize where you use the Axiom of Choice.

4. Evaluate

$$\lim_{n\to\infty}\int_{(-\infty,\infty)}\left(\frac{510+1488(\sin x)^{12222}}{1998}\right)^n\frac{1}{1+x^2}\,dx$$

Justify your conclusion.

5.a. Define what it means for a function f to be of bounded variation on $[a,b]$.

b. Prove that a function f is of bounded variation on $[a,b]$ if and only if f is the difference of two monotone real valued functions on $[a,b]$.

c. Find a continuous function on $[0,1]$ which is not a function of bounded variation, and prove that this function is not of bounded variation.

6.a. Give an example of a sequence f_n of *continuous* functions converging pointwise to a *continuous* function f on $[0,1]$ such that

$$\int_{[0,1]} f_n \not\to \int_{[0,1]} f$$

b. State the Bounded Convergence Theorem, and explain why the example in part a does not give a counterexample to this theorem.

7.a. Define what it means for an extended real-valued function $f(x)$ to be (Lebesgue) measurable. This definition is connected to four equivalent statements; prove the equivalences.
b. Use the definition in part a to prove or disprove the following statement. If f and g are real-valued measurable functions, then $\{x : f(x) \leq g(x)\}$ is measurable.