

Preliminary Examination

Real Analysis

January 1998

Do all of problems 1-2, and as many of the remaining problems as possible.

1. State the following theorems.

- (a) Monotone Convergence Theorem
- (b) Fatou's Lemma
- (c) Dominated Convergence Theorem
- (d) Hölder's Inequality
- (e) Fubini's Theorem
- (f) Egoroff's Theorem

2. Prove 1(a), (b), and (c) in that order.

3. Let λ be Lebesgue measure and μ be counting measure both regarded as Borel measures on $I = [0, 1]$. Let Δ be the diagonal in $I \times I$; $\Delta = \{(x, y) | x = y\}$.

(a) Show that Δ is measurable (with respect to the product measure on Borel subsets of $I \times I$).

(b) Let f be the characteristic function in Δ . Compute the integrals: $\int_I (\int_1 f d\lambda) d\mu$,
 $\int_I (\int_I f(d\mu) d\lambda)$ and $\int_{I \times I} f d\mu \times d\lambda$.

(c) Reconcile with Fubini's Theorem.

4. Let m be Lebesgue measure on $X = [0, 1]$, and let μ be a measure on the Lebesgue sets with $\mu(X) = 1$, and $\mu \sim m$ (i.e. μ and m have the same sets of measure zero).

Prove there exists a measurable set A such that $\mu(A) = 1/2$.

5. Prove that the product of two measurable functions is a measurable function.

6. (a) Define bounded variation of a function f on $[a, b]$.
- (b) Show that if f is a function of bounded variation on $[a, b]$, then f has at most a countable number of discontinuities.
7. Let f_n be a sequence of measurable functions such that $f_n(x) \rightarrow f(x)$ almost everywhere, and suppose that $\sup \int_0^1 |f_n(x)| dx < \infty$.
- (a) Show that f is measurable and that $\int_0^1 |f(x)| dx < \infty$.
- (b) Does $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx$?
8. Give an example of a sequence f_n of continuous functions on $[0, 1]$ converging pointwise to a continuous function f on $[0, 1]$ such that $\int f_n \notin \int f$.
9. Define a sequence of measures (μ_n) on the Lebesgue measurable subsets of $[0, 1]$ by $\mu_n(A) = \int_0^1 I_A(x) n x^{n-1} dx$, $n = 1, 2, 3, \dots$.
- (a) Verify that $\lim_{n \rightarrow \infty} \mu_n([a, b]) = 0$ if $0 < a < b < 1$.
- (b) Suppose $0 < a_k < b_k < 1$ and $[a_k, b_k]$ are disjoint, $k = 1, 2, \dots, r$. Verify $\lim_{n \rightarrow \infty} \mu_n\left(\bigcup_{k=1}^r [a_k, b_k]\right) = 0$
- (c) Suppose $[a_k, b_k]$, $k = 1, 2, 3, \dots$, are all disjoint. Does $\lim_{n \rightarrow \infty} \mu_n\left(\bigcup_{k=1}^{\infty} [a_k, b_k]\right)$ exist? Give reasoning for your answer.
10. Let A be a set of positive Lebesgue measure. Show that the set $A + A = \{a + b : a, b \text{ in } A\}$ contains an open interval.