University at Albany

Department of Mathematics and Statistics

Ph.D. Preliminary Examination

Real Analysis

Monday, June 3, 1996

Do eight of the following ten problems. If you do more, please indicate which eight you wish to be graded.

1. Give precise statements of the following:
   - A. The Monotone Convergence Theorem
   - B. Egoroff's Theorem
   - C. The Radon-Nikodym Theorem (include a definition of absolute continuity for measures)
   - D. Fubini's theorem
   - E. Hölder's Inequality
   - F. Fatou's Lemma

2. Using only properties of Lebesgue measure and the definition of Lebesgue integral prove the Monotone Convergence Theorem.

3. A. Define what it means for a function on [0,1] to be Absolutely Continuous.

   B. Prove that an absolutely continuous function is continuous.

   C. Prove that an absolutely continuous function is of bounded variation.

4. If f and g are measurable functions on [0,1] prove that $\{x : f(x) < g(x)\}$ is measurable.

5. If S is an infinite subset of R prove that S contains a countable dense subset.

6. If f is a differentiable function on $(-\infty, \infty)$ with both f and f' in $L(-\infty, \infty)$ show that $\displaystyle\int_{-\infty}^{\infty} f'(x)\, dx = 0$.

7. Prove that if f and g are positive, continuous functions on $(-\infty, \infty)$ which are periodic of period 1 that

$$\lim_{n\to\infty} \int_0^1 f(x)\ g(nx)\ dx = \int_0^1 f(x)\ dx \cdot \int_0^1 g(x)\ dx$$

8. Give an example of a closed bounded subset of $L^2[0,1]$ which is not compact. Prove your answer.

9. Construct a non-Lebesgue-measurable subset of the real numbers.

10. Let λ denote Lebesgue measure on $I = [0,1]$ and μ counting measure both regarded as Borel measures on I. The diagonal:

$$\Delta = \{(x,y) : x = y, (x,y) \in I \times I\}\ .$$

   A. Show that Δ is measurable with respect to product measure, $\lambda \times \mu$ on the Borel sets of $I \times I$.

   B. Let $f(x,y) = \chi_\Delta(x,y)$, the characteristic function of Δ. Compute $\int_I \left(\int_I f\ d\lambda \right)\ d\mu$, $\int_I \left(\int_I f\ d\mu \right)\ d\lambda$, and $\int_{I\times I} f\ d\lambda \times d\mu$.

   C. Reconcile part B with Fubini's Theorem.