Department of Mathematics and Statistics

Ph.D. Program

Preliminary Examination in Real Analysis

Friday, January 20, 1995

Do all eight problems.

1. State the following theorems:
   Fatou's lemma,
   Lebesgue dominated convergence theorem,
   Fubini's theorem,
   Egoroff's theorem,
   Hölder's inequality,
   The Radon-Nikodym theorem.

2. State and prove the Monotone convergence theorem (without relying on any of the other convergence theorems).

3. Prove that the sum of two measurable functions is a measurable function.

4. Show that if f is a monotone function on $[a,b]$ then f has at most a countable number of discontinuities.

5. Let f_n be a sequence of measurable functions such that $f_n(x) \to f(x)$ almost everywhere, and suppose that $\sup \int_0^1 |f_n(x)|dx < \infty$.

   (a) Show that f is measurable and that $\int_0^1 |f(x)|dx < \infty$.

   (b) Does $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$?

6. Show how to construct a non-constant continuous function on [0,1] which is differentiable at each rational point and such that for every rational number x,
$$f'(x) = 0 \ .$$

7. Let $f(x)$ be a measurable function on $[0,\infty)$ such that
$$\int_0^\infty [f(x)]^n dx = c \quad \text{for } n = 2, 3, 4 \ .$$
   Show that $f(x) = \chi_A(x)$ almost everywhere for some measurable set $A \subseteq [0,\infty)$.

8. Show that of $f \in L^1(X,\mu)$ then
$$\int_0^\infty \mu\{x : |f(x)| > t\} \ dt = \int_X |f(x)| \ d\mu(x) \ .$$