Ph.D. Preliminary Exam in Real Analysis

August 1994

Do problems 1 and 2 and as many of the remaining as time permits. Strive for complete solutions.

1. State the following:

   (a) Fatou's Lemma

   (b) Monotone Convergence Theorem

   (c) Lebesgue Dominated Convergence Theorem

   (d) Hölders Inequality

   (e) Egoroff's Theorem

   (f) Fubini's Theorem

2. Starting with basic principles, prove 1.(a), (b) and (c) in any order.

3. (a) Prove that the sum of two Lebesgue measurable functions is measurable.

   (b) Give an example of a function which is in $L^3([0,1])$ but not in $L^2([0,1])$ (Lebesgue) and justify your answer.

4. Evaluate $\lim_{n\to\infty} \int_0^{\pi/2} ne^{-x^2} \cos x \sin^{n-1} x \, dx$, justifying your computations.

5. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let ν be a measure such that $\nu << \mu$

   (a) State the Radon-Nikodym Theorem in this context.

   (b) If f is a nonnegative measurable function, show that $\int f d\nu = \int f[\frac{d\nu}{d\mu}] d\mu$ where $[\,]$ denotes the $R - N$ derivative.

6. (a) Let $f : [0,1] \to \mathbf{R}$ be given by $f(0) = 0$, $f(x) = x^{1/2}$ for $x > 0$. Show by direct computation that f is absolutely continuous.

   (b) Let $g : [0,1] \to \mathbf{R}$ be given by $g(0) = 0$, $g(x) = x \sin \frac{1}{x}$ for $x > 0$. Is g of bounded variation? (Justify completely.)

(c) Is every continuous function on $[0,1]$ absolutely continuous? (Justify in reasonable detail.)

7. Let $\mu =$ Lebesgue measure on $[0,1]$ and recall that, for a measurable function f, $\|f\|_\infty$ is defined to be $\inf\{m : \mu\{t \| f(t)| > m\} = 0\}$. Prove that if $\|f\|_\infty < \infty$, then f is in L^p for all $p \geq 1$ and that $\lim\limits_{p \to \infty} \|f\|_p = \|f\|_\infty$.

8. Let $X = Y = [0,1]$ and $\mu = \nu =$ Lebesgue measure.

   (a) Prove that each Borel set in $X \times Y$ is measurable with respect to $\mu \times \nu$.

   (b) Let h and g be μ-integrable on [0,1] and define f on $X \times Y$ by $f(x,y) = h(x) \cdot g(y)$. Prove that f is integrable and establish a formula for $\int_{X \times Y} f d(\mu \times \nu)$ in terms of h and g.

9. (a) Show that the (improper) Riemann integral of a function may exist, on say $[0,\infty)$, although the function may not be Lebesgue integrable on $[0,\infty)$.

   (b) Let $f \geq 0$ be Lebesgue integrable on $\mathbf{R}$. Show that the function $F(x) = \int_{-\infty}^{x} f$ is continuous.