Preliminary Examination

Real Analysis

September 1993

Do all of problems 1-2, and as many of the remaining problems as possible.

1. State the following theorems.

   (a) Monotone Convergence Theorem

   (b) Fatou's Lemma

   (c) Dominated Convergence Theorem

   (d) Hölder's Inequality

   (e) Fubini's Theorem

   (f) Egoroff's Theorem

2. Prove 1(a), (b), and (c) in that order.

3. Give an example of a sequence f_n of continuous functions on [0,1] converging pointwise to a continuous function f on [0,1] such that $\int f_n \not\to \int f$.

4. Define a sequence of measures (μ_n) on the Lebesgue measurable subsets of [0,1] by $\mu_n(A) = \int_0^1 I_A(x)\ \frac{1}{n}\ x^{1/n\ -\ 1} dx,\ n = 1, 2, 3, \ldots$.

   (a) Verify that $\lim_{n\to\infty} \mu_n([a, b]) = 0$ if $0 < a < b < 1$.

   (b) Suppose $0 < a_k < b_k < 1$ and $[a_k, b_k]$ are disjoint, $k = 1, 2, \ldots, r$. Verify $\lim_{n\to\infty} \mu_n(\bigcup_{k=1}^{r} [a_k, b_k]) = 0$.

   (c) Suppose $[a_k, b_k]$, $k = 1, 2, 3, \ldots$, are all disjoint. Does $\lim_{n\to\infty} \mu_n(\bigcup_{k=1}^{\infty} [a_k, b_k])$ exist? Give reasoning for your answer.

5. Let m be Lebesgue measure on $X = [0,1]$, and let μ be a measure on the Lebesgue sets with $\mu(X) = 1$, and $\mu \sim m$ (i.e. μ and m have the same sets of measure zero). Prove there exists a measurable set A such that $\mu(A) = 1/2$.

6. Let $\mathcal{M}$ be a class of subsets of a set X. $\mathcal{M}$ is a monotone class if $M_n \in \mathcal{M}$ and either $M_n \uparrow M$ or $M_n \downarrow M$ implies $M \in \mathcal{M}$. Show that if a monotone class $\mathcal{M}$ contains an algebra $\mathcal{A}$, then $\mathcal{M}$ contains the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$.

7. Construct a sequence of non-negative functions f_n on [0,1] such that $\displaystyle\int_0^1 f_n(x)dx = 1$ for all n, $\displaystyle\lim_{n\to\infty} f_n(x) = 0$ a.e., and for each $g \in C[0,1]$, $\displaystyle\lim_{n\to\infty}\int_0^1 f_n(x)\ g(x)\ dx = (g(1/5) + g(2/5) + g(3/5))/3$. (Be sure to verify the last statement.)

8. Let E be a subset of [0,1] of Lebesgue measure 0. Show that there is an absolutely continuous function f whose derivative $f'(x) = +\infty$ for each $x \in E$.

9. Let $\{f_n\}_{n=1}^{\infty}$ and f be non-decreasing functions on [0,1] and suppose the f_n converge in measure to f. Prove that $\lim f_n(x) = f(x)$ at every point, x, where f is continuous.

10. Let $f(x) \geq 0$ for each $x \in [0,1]$, f bounded, and suppose that $E = \{(x,y) : 0 \leq y \leq f(x)\}$ restricted to $0 \leq x \leq 1$ is a measurable subset of the plane. Prove that f is Lebesgue integrable over [0,1].