Department of Mathematics and Statistics

Ph.D. Program

Preliminary Examination in Real Analysis

Tuesday, June 8, 1993

Directions: Do all problems except possibly 7, which is an extra credit problem. The last question is meant to be a longer essay-type question and you should take more time and write more on it. It is suggested that you do this problem last.

1. Let (X, B, μ) be a measure space, and let $\{f_n\}$ be a sequence of measurable functions from X into $\mathbf{R}$, such that $f_n \geq 0$, and $f_n \geq f_{n+1}$, $n = 1, 2, \ldots$ Suppose $\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = 0$. Show that $\lim_{n\to\infty} f_n = 0$ a.e. on X.

2. Define the Cantor ternary set.

   A. Prove that it has power of the continuum.

   B. Prove that it has Lebesgue measure 0.

   C. Prove that its characteristic function is Riemann integrable on [0,1].

3. Let $A_1, A_2, \ldots, A_n, \ldots$ be a sequence of points of $R \times R$. Suppose there is a number $0 < r < 1$ such that the distance from A_n to A_{n+1} is r times the distance from A_{n-1} to A_n. Prove that the sequence of points is convergent.

4. Let f be continuous on the entire real line. For each positive integer n, let g_n be the characteristic function of $[0,\ 1/n]$ and let f_n be defined by

$$f_n(x) = n \int_{-\infty}^{\infty} g_n(x-t) f(t)\, dt\ .$$

   A. Show that $f_n \to f$ pointwise.

   B. Prove or disprove: f_n converges to f uniformly.

5. In $R \times R$ let C denote the family of closed rectangles with sides parallel to the axes and let D denote the family of closed discs (i.e. $E \in D$ if and only if there is a center (a, b) and a radius $0 < r < \infty$ such that $E = \{(x, y)\} : (x-a)^2 + (y-b)^2 \leq r\}$. Prove that the σ-ring of sets generated by C coincides with the σ-ring generated by D.

6. In this problem all L^p spaces are formed on the real line with Lebesgue measure.

A. If $0 < p < q < r \leq \infty$ show that $L^p \cap L^r \subset L^q$ and that for $f \in L^p \cap L^r$ one has

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$$

where λ satisfies $\dfrac{1}{q} = \dfrac{\lambda}{p} + \dfrac{1-\lambda}{r}$.

B. Show, by giving appropriate examples, that for

$$0 < p < q \leq \infty \ ,$$

there is no inclusion of either the form $L^p \subset L^q$ or $L^q \subset L^p$.

7. (Extra credit) Let $r_1, r_2, \ldots, r_n, \ldots$ be an enumeration of the rational numbers in [0,1]. Let

$$f(x) = \sum_{\{n : x > r_n\}} \frac{1}{2^n} \ .$$

Compute $\int_0^1 f(x)\ dx$ in terms of $\{r_n\}$.

8. A. Write down the axioms for a measure space (X, B, μ).

B. Use your axioms to define the Lebesgue integral,

$$\int_X f(x)\ d\mu(x) \ ,$$

(for simplicity, it will be enough to do this problem only for non-negative functions.)

C. Assume $0 < \mu(X) < \infty$. Prove, using the axioms of part A and the definition of part B, the following lemma (from which most of the convergence theorems follow as an easy consequence).

Lemma: Let $\{f_n(x)\}$ be a sequence of integrable functions. Suppose there is a constant M such that for all $x \in X$ and all n,

$$0 \leq f_n(x) \leq M < \infty$$

and there is a function $g(x)$ such that $\lim_{n\to\infty} f_n(x) = g(x)$ almost everywhere. Then $g(x)$ is integrable and

$$\lim_{n\to\infty} \int_X f_n(x)\ d\mu(x) = \int_X g(x)\ d\mu(x) \ .$$

(The idea of the problem is to prove a convergence theorem from basic principles; in particular, Lebesgue dominated convergence or Fatou's lemma should not be used unless you have proved them from basic principles as part of this essay.)