Department of Mathematics and Statistics
Ph.D. Preliminary Examination in Real Analysis
Saturday, January 23, 1993

Do all 8 problems.

1. State the following theorems.

   Fatou's lemma

   Lebesgue Dominated Convergence Theorem

   Lebesgue Monotone Convergence Theorem

   Egoroff's Theorem

   Minkowski's Inequality

   The Radon-Nikodym Theorem

2. Prove Fatou's Lemma from basic principles.

3. Let E be the subset of $[0,1]$ such that $x \in E$ if and only if there is only one 9 in the decimal expansion of E. Prove that E has Lebesgue measure 0.

4. Calculate
$$\lim_{h\to\infty} \int_0^1 \frac{h^{3/2}x^{3/2}}{1+h^2x^2} dx \ .$$
   Justify your calculation.

5. Let μ be a finite measure on the Borel sets of $(-\infty, \infty)$. Let
$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\mu(t) \ .$$
   Prove or give a counterexample: $f(x)$ is uniformly continuous on $(-\infty, \infty)$.

6. Let $f(x) \geq 0$ be a function $[0,1]$ and let $E = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq f(x)\}$. Prove that if E is a 2-dimensional Lebesgue measurable set than f is a Lebesgue measurable function.

7. Let $f(x)$ be a Lebesgue integrable function such that $\int_0^1 f(x)x^n dx = 0$ for all $n \geq 2$. Prove or give a counterexample: $f(x) = 0$ almost everywhere.

8. Let A and B be Lebesgue measurable sets of finite non-zero measure. Let $\varphi(x) = |A \cap (B + x)|$ where absolute value denotes Lebesgue measure and $B + x = \{y : y = b + x \text{ for some } b \in B\}$. Prove or give a counterexample: $\varphi(x)$ is continuous.