Ph.D Preliminarly Exam

Real Analysis

August 2008

Do as many problems as you can and as completely as possible. Unless otherwise stated, all problems refer to Lebesgue measure and integration.

1. (a) State the Monotone Convergence Theorem, The Dominated Convergence Theorem, Fatou's Lemma and prove one of them.

   (b) Give an example of a sequence $\{f_n\}$ of continuous functions on $[0,1]$ converging pointwise to a continuous function, f, but $\int_0^1 f_n \not\to \int_0^1 f$.

2. Define what it means for $f : \mathbf{R} \to \mathbf{R}$ to be measurable. Prove in detail that if $f : \mathbf{R} \to \mathbf{R}$ is measurable and $g : \mathbf{R} \to \mathbf{R}$ is continuous, then $g \circ f$ is measurable.

3. Compute $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{n \cos x \sin^n x}{1+x} dx$, justifying each step.

4. Suppose f is integrable on $[0,1]$. Prove that $\lim_{n\to\infty} \int_0^1 f(x) \cos nx \; dx = 0$. Hint: You can use, without proof, appropriate results about approximation of integrable functions and measurable sets.

5. (a) Define $\|f\|_p$ and L^p on $[0,1]$.

   (b) Prove that if $0 < p < q < \infty$ and $f \in L^q$ then $\|f\|_p \leq \|f\|_q$ (on $[0,1]$).

   (c) Give an example of a function f on $[0,1]$ that is in L^p for $1 \leq p < 4$, but not in L^4.

6. Define absolute continuity (A.C.) on $[0,1]$. Prove that if f is A.C. and monotone increasing on $[0,1]$ and E is a set of measure zero, then $f(E)$ has measure zero.

7. State what it means for $f : [0,1] \to \mathbf{R}$ to be of bounded variation (B.V.). Prove that if f is integrable on $[0,1]$, then $F(x) = \int_0^x f$ is B.V.

8. Let λ be a Lebesgue measure and μ be counting measure both regarded as Borel measures on $I = [0, 1]$. Let Δ be diagonal in $I \times I$; $\Delta = \{(x, y) | x = y\}$.

   (a) Show that Δ is measurable (with respect to the product measure on Borel subsets of $I \times I$.

   (b) Let f be the characteristic function in Δ. Compute the integrals: $\int_I (\int_1 f d\lambda) d\mu$, $\int_I (\int_I f(d\mu) d\lambda)$ and $\int_{I \times I} f d\mu \times d\lambda$.

   (c) Reconcile with Fubini's Theorem.