

The University at Albany
Department of Mathematics and Statistics
Ph. D. Program
Preliminary Examination in Real Analysis
January 2008

Part I

Problem 1. State the following

- a. Egorov's Theorem.
- b. Fatou's Lemma.
- c. Monotone Convergence Theorem.
- d. Lebesgue Dominated Convergence Theorem.
- e. Hölders Inequality.

Problem 2. Prove (b) and (d).

PART II

Problem 3. Prove that the product of two measurable functions is measurable.

Problem 4. Give an example of a sequence of continuous functions f_n with domain $[0, 1]$ converging pointwise to a continuous function f also with domain $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} \int f_n dx \neq \int f dx.$$

Problem 5. Show that if $f \in L^p(\mathbb{R})$, then

$$\int |f(x)|^p d\lambda(x) = \int_0^\infty p t^{p-1} \lambda(\{x : |f(x)| > t\}) dt,$$

where λ denotes the Lebesgue measure on \mathbb{R} . Justify all your steps.

Hint: one way to prove this is to write the right hand side as a double integral.

Problem 6. Let f be an integrable function on \mathbb{R} . Show that given any $\epsilon > 0$, there exists a simple function g on \mathbb{R} such that

$$\int |f(x) - g(x)| dx < \epsilon.$$

Problem 7. Let λ be the Lebesgue measure on $[0, 1]$. Recall that if f is a function with domain $[0, 1]$, then

$$\|f\|_\infty = \text{ess sup}_{x \in [0, 1]} |f(x)| = \inf\{M : \lambda(\{x \in [0, 1] : |f(x)| > M\}) = 0\},$$

and if $0 < p < \infty$,

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{1/p}.$$

Prove that if f is a measurable function with domain $[0, 1]$ and $\|f\|_\infty < \infty$, then

a. $f \in L^p([0, 1])$ for all $p > 0$ and $\|f\|_p \leq \|f\|_\infty$,

b. if $1 \leq p < q < \infty$ then $\|f\|_p \leq \|f\|_q$,

c.

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$