Real Analysis Prelim. (January, 2007)

PART I: Do all of the following problems.

1. Show that the interval $[0,\ 1]$ is uncountable.

2. Prove that the set of continuous functions on $[0,\ 1]$ (denoted by $C[0,\ 1]$) with uniform metric is a complete metric space.

3. State the following theorems:

a, Fatou's Lemma;
b, Lebesgue Dominated Convergence Theorem;
c, Baire's Category Theorem;
d, Radon-Nikodym Theorem.

4. Prove 3(b).

PART II: Do at least four of the following problems.

5. Prove 3(c).

6. Let λ be the Lebesgue measure on $[0,\ 1]$, and let f be defined on $[0,\ 1]$ such that $f(x) = 0$ if x is irrational and $f(x) = 1/n$ if $x = m/n$ where m and n are coprime nonnegative integers. Show that f is Riemann integerable.

7. Let f be an integerable function on $[0,\ 1]$. Show that if

$$\int_0^c f(x)dx = 0$$

for every $c \in [0,\ 1]$, then $f = 0$ almost everywhere on $[0,\ 1]$.

8. Let $\{f_n\}$ be a sequence in $L^2(0,\ 1)$ that converges to f in norm (i.e. $\lim_{n\to\infty} \|f_n - f\| = 0$). Show that f_n converges in measure to f.

9. Let μ be a Borel measure on $[1,\ 2]$ such that $\mu(a,\ b) = Ln(b) - Ln(a)$ for every $a,\ b \in [1,\ 2]$. Show that $\mu << \lambda$, where λ is the Lebesgue measure, and find $d\mu/d\lambda$.

10. Let f be in $L^1(\mathbf{R}) \cap \mathbf{L^2}(\mathbf{R})$.

(a) Show that $f \in L^p(\mathbf{R})$ for every $1 \le p \le 2$.

(b) Show that

$$\lim_{p\to 1^+} \|f\|_p = \|f\|_1.$$