Department of Mathematics and Statistics
Preliminary Examination in Probability
September 2006

Do all 6 problems.

1. Suppose a machine flips a fair coin indefinitely and at each flip it records whether it came up head of tail. We may start looking at the results at the n^{th} flip of the coin and we wonder how many heads in a row we will see. Let $l_n =$ the number of consecutive heads we see starting from the n^{th} toss, that is $l_n = k$ if the tosses numbered $n, n+1, ..., n+k-1$ are all heads and the $n+k$ toss is a tail.

(a) Find $P(l_n \geq n$ for infinitely many n's)
(b) Find $P(l_n = 1$ for infinitely many n's)

2. Let $X_1, X_2, ..., X_n, ..$ be iid.

(a) Prove $E|X_1| < \infty \iff P(|X_n| > n$ for infinitely many n's $) = 0$
(b) State the Strong Law of Large Numbers.
(c) Explain why the Strong Law of Large Numbers fails when $E|X_1| = \infty$

3. Show that if $Y_n \to Y$ in distribution and $X_n \to c$ in probability where $c > 0$ is a constant, then $Y_n/X_n \to Y/c$ in distribution.

4. Let $X_1, X_2, ...$ be iid with $E(X_1) = 0$ and $E(|X_1|^2) = 4$. Find the limit in distribution of

$$\sqrt{n}\,\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_1^2}$$

5. Let $Y_1, Y_2, ..., Y_n, ...$ be independent Poisson random variables with parameter λ.

(a) Find the characteristic function of Y_n
(b) Find the characteristic functions of $S_n = Y_1 + Y_2 + ... + Y_n$
(c) Find the characteristic function of $(S_n - n\lambda)/\sqrt{n\lambda}$ and its limit (with justification) as $n \to \infty$.
(d) What does the limit in part (c) tell you about the limiting distribution of $(S_n - n\lambda)/\sqrt{n\lambda}$?

6. Consider the interval $[0,1]$. Let β_n the $\sigma-$algebra generated by the dyadic intervals of length 2^{-n}, that is $[0, 1/2^n), [1/2^n, 2/2^n), [2/2^n, 3/2^n), ..., [2^{n-1}/2^n, 1]$. Let f be an integrable function on $[0,1]$.

(a) Give an expression for $E(f|\beta_n)$. (Try $n = 1, n = 2$ then derive the general expression)
(b) Show that $E(f|\beta_n)$ is a martingale.
(c) Find $\lim_{n\to\infty} E(f|\beta_n)$