Real Analysis Preliminary Exam **August 26, 2005**

Name: **ID:**

1. Do the following things.

(a) State the Lebesgue Dominated Convergence Theorem.

(b) State Fatou's lemma, and give an example to show that the equality may not hold.

(c) State the Stone-Weierstrass Theorem.

(d) State the Minkowski's Inequality.

2. State and prove the Egoroff's Theorem.

3. Describe how the Lebesgue measure on $\mathbf{R}$ is constructed.

4. Let $f(x)$ and $g(x)$ be two functions on $\mathbf{R}$ such that f is continuous and g is measurable. Prove that the composition $f(g(x))$ is a measurable function.

5. Show that if a monotone sequence $\{f_n\}$ of continuous functions on $[0, 1]$ converges pointwise to a continuous function, then $\{f_n\}$ converges uniformly.

6. Show that the set of irrational numbers is not a countable union of closed subsets of $\mathbf{R}$. (Hint: Use Baire's category theorem.)

7. Let f be a Lebesgue integrable function over $[0,\ 1]$ and define $F(x) = \int_{[0,\ x]} f d\lambda$. Show that F is absolutely continuous.

8. Assume $f \in L^1(\mathbf{R})$ and $E_\epsilon = \{x :\ |f(x)| \geq \epsilon > 0\}$. Show that

$$\lambda(E_\epsilon) \leq \frac{\|f\|}{\epsilon}.$$

9. Assume $0 < p < q \leq \infty$. If f is in $L^q([0,\ 1])$, then show that $\|f\|_p \leq \|f\|_q$.

10. Let $f(x,y)$ be a function on the unit square $D = [0,1] \times [0,1]$ such that for every $x \in [0,1]$ $f(x,y)$ is an integrable function of y, and

$$\int_0^1 f(x,y)dx = g(x)$$

is a continuous function of x. True, or false: This implies that f is integrable on D and

$$\int_D f(x,y)dxdy = \int_0^1 g(x)dx\ ?$$