

Real Analysis Preliminary Exam

August 29, 2003

Do all of these problems.

- (a) State the Lebesgue Dominated Convergence Theorem, Fatou's Lemma, the Lebesgue Monotone Convergence Theorem.

(b) Pick one of the above and prove it from scratch, not using either of the other two.
- (a) Define what it means for a function $f : \mathbf{R} \rightarrow \mathbf{R}$ to be measurable. (You can include several equivalent definitions if you wish.)

(b) Assuming what you need to know about Lebesgue measurable sets in \mathbf{R} , prove that, if $\{f_n\}$ is a sequence of measurable functions, then $\limsup f_n$ is also measurable.
- (a) Define what it means for a function on $[0, 1]$ to be absolutely continuous.

(b) Prove that an absolutely continuous function is continuous.

(c) Prove that an absolutely continuous function is of bounded variation.
4. Evaluate $\lim_{n \rightarrow \infty} \int_0^{\pi/2} n e^{x^3} \cos(x) \sin^{(n-1)}(x) dx$, justifying your steps.
- (a) Define $L^p([0, 1])$ and, without proof, summarize its important properties.

(b) Give an example of a function that is in $L^p([0, 1])$ for $1 \leq p < 4$ but not in $L^4([0, 1])$. Justify your assertions.
6. Suppose f is Lebesgue integrable on $[0, 1]$. prove that $\lim_{n \rightarrow \infty} \int_0^1 f(x) \cos(nx) dx = 0$.
Hint: what is the simplest integrable function you can think of?

7. Let λ be Lebesgue measure and μ be counting measure both regarded as Borel measures on $I = [0, 1]$. Let Δ be the diagonal in $I \times I$; $\Delta = \{(x, y) | x = y\}$.

(a) Show that Δ is measurable (with respect to the product measure on Borel subsets of $I \times I$).

(b) Let f be the characteristic function in Δ . Compute the integrals: $\int_I (\int_I f \, d\lambda) \, d\mu$, and $\int_I (\int_I f \, d\mu) \, d\lambda$.

(c) Explain why the result of part (b) does not contradict Fubini's Theorem. As part of the explanation, you should compute the double integral $\int_{I \times I} f \, d\mu \times d\lambda$.