Preliminary Examination
Real Analysis
Fall 2001

Do all 8 problems.

(1) State the following theorems:
   (a) Dominated convergence theorem.
   (b) Monotone convergence theorem.
   (c) Fatou's lemma.
   (d) Egoroff Theorem.
   (e) Radon–Nikodym theorem.
   (f) Fubini's theorem.

(2) Prove the Dominated Convergence Theorem.

(3) Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$. Let $f(x) = \liminf_{n\to\infty} f_n(x)$. Show that f is measurable.

(4) Let f be a continuous function on the interval $[a, b]$. Prove or disprove:
   (a) if f is absolutely continuous then f has bounded variation.
   (b) if f has bounded variation then f is absolutely continuous.

(5) Let $\{r_n\}$ be a sequence of real numbers. Show

$$\lim_{n\to\infty} \int_0^\infty e^{-x} \sin^n(x + r_n)\ dx = 0$$

(6) Let $f \in L^1(0, 2\pi)$. Prove that

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\ e^{-inx}\ dx = 0$$

(7) (a) Show that if f, g are measurable in $\mathbb{R}$ then $(x, y) \to f(x-y)g(y)$ is measurable in $\mathbb{R}^2$.
   (b) Show that

$$f * g(x) = \int f(x - y)g(x)dx$$

   is an integrable function and $\|f * g\|_1 \le \|f\|_1\ \|g\|_1$.
   (c) Show that if g is continuous with compact support, then $f * g$ is uniformly continuous.

(8) Let $\{a_n\}$ be a sequence of positive reals such that $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$. Show there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_n = 1$ or -1 such that

$$\limsup_{N\to\infty} \sum_{n=1}^N \epsilon_n a_n = +\infty$$

$$\liminf_{N\to\infty} \sum_{n=1}^N \epsilon_n a_n = -\infty$$