From: M Hildebrand ¡martinhi@csc.albany.edu¿ Date sent: Mon, 4 Jun 2001 10:30:57 -0400 (EDT) ja984@csc.albany.edu Subject: Today's Real Analysis Prelim (TeX file)

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Preliminary Examination

Real Analysis

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Do as many problems as possible!

1. State the following theorems.

a. Monotone Convergence Theorem

b. Fatou's Lemma

c. Dominated Convergence Theorem

d. Hölder's Inequality

2. Let E_n be a sequence of measurable sets such that $E_{n+1} \subseteq E_n$ for each n. Let m be Lebesgue measure.

a. Give an example where

$$m(\cap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} m(E_n).$$

b. Impose an additional condition that forces

$$m(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n),$$

and prove the result with the condition. Your proof should begin with basic properties of measure theory.

3. Let X be the interval $[0,1]$, $\mathcal{B}$ be the collection of Borel sets of X, and μ be Lebesgue measure. In $X \times X$, is the diagonal $D := \{(x,x) | x \in X\}$ measurable under product measure? Justify your response.

4. Show that in $[0,1)$ with Lebesgue measure, there exists a nonmeasurable set. Emphasize where you use the Axiom of Choice.

5. Let

$$f(i,j) = \begin{cases} 0 & \text{if } j > i \\ 2/(i+1) & \text{if } j \leq i \text{ and } i \text{ is odd} \\ -2/i & \text{if } j \leq i,\ i \text{ is even, and } j \text{ is odd} \\ -2/(i+2) & \text{if } j \leq i,\ i \text{ is even, and } j \text{ is even} \end{cases}$$

a. Compute

$$\sum_{i=1}^{\infty} f(i,j)$$

for each j and

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f(i,j).$$

(Assume the standard calculus definitions of convergent series.)

b. Can one use Fubini's Theorem (on $\mathbf{Z}^+ \times \mathbf{Z}^+$ with counting measure on $\mathbf{Z}^+$) to claim

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(i,j) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f(i,j)?$$

Justify carefully!

6. True or false. Justify your answer with a proof or counterexample.

a. Suppose f_n is a nonnegative measurable function with $f_n(x) \le 1/(x^2+1)$. Suppose $f_n \to f$ almost surely where f is a measurable function on the real numbers. Then $\int f_n \to \int f$.

b. Suppose f_n is a nonegative function on the positive integers with $f_n(j) \le 2001$ for each positive integer j. Suppose $f_n(j) \to f(j)$ for each positive integer j. Suppose that

$$\sum_{j=1}^{\infty} f_n(j)$$

is finite for each n and

$$\sum_{j=1}^{\infty} f(j)$$

is finite. Then

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} f_n(j) = \sum_{j=1}^{\infty} f(j).$$

7. Describe a continuous function on $[0,1]$ such that $f(0)=0$, $f(1)=1$, and $f'(x)=0$ a.e. Justify your answer.

8.a. Give a definition describing what it means for a function f on $[a, b]$ to be of bounded variation.

b. Prove that a function f on $[a, b]$ is of bounded variation if and only if f is the difference of two monotone functions on $[a, b]$.