The University at Albany

Department of Mathematics and Statistics

Ph.D. Program

Preliminary Examination in Real Analysis

Thursday, August 31, 2000

Part I

1. State the following theorems:

   A. The Lebesgue Dominated Convergence Theorem,

   B. Fatou's Lemma,

   C. Egoroff's Theorem,

   D. The Fubini Theorem,

   E. Hölder's Inequality.

2. Given the properties of measure theory, give a definition of the Lebesgue integral.

Part II

Do 6 of the following 8 problems:

3. Prove if E and F are subsets of the real line with positive Lebesgue measure and, if

$$E + F = \{x : x = y + z \text{ with } y \in E \text{ and } z \in F\}$$

   then $E + F$ contains a non-empty open interval.

4. A. Construct the Cantor ternary set, C.

   B. Prove that C has Lebesgue measure 0.

   C. Prove that the characteristic function of C is Riemann integrable.

5. Prove there exists a non-measurable subset of the real line.

6. Let $E \subseteq [0, 1)$ where $x \in E$ if x has no 9's in its decimal expansion. Prove that E has Lebesgue measure 0.

7. If f is a measurable function on $[a, b]$, $a < b$ and we define

$$\|f\|_\infty = \inf\{M \geq 0 : |E_M| = 0\}$$

where $E_M = \{x : |f(x)| > M\}$. $\|f\|_\infty$ is called the essential supremum of f.
Prove that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p = \lim_{p\to\infty} \left[\int_a^b |f(x)|^p dx\right]^{1/p}$.

8. Prove that the infinite sum

$$\sum_{n=0}^{\infty} \int_0^{\pi/3} (1 - \sqrt{\sin x})^n \cos x \; dx$$

has a finite limit and find its value.

9. Prove if f_n is a sequence of Lebesgue integrable functions with $\|f_n\|_1 = 1$ for $n = 1, 2, 3, \ldots$ and if the measure $\{\text{support of } f_n\} \to 0$ as $n \to \infty$ then for all $p > 1$ we have

$$\|f_n\|_p \to \infty \quad \text{as } n \to \infty \; .$$

10. Prove that if f is a differentiable function on $(-\infty, \infty)$ such that f and f' are both in $L^1(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f'(x) dx = 0 \; .$$