Real Analysis Prelim

June 2000

Complete as many problems as possible.

1. (a) State the Lebesgue Dominated Convergence Theorem, Fatou's Lemma, the Lebesgue Monotone Convergence Theorem.

   (b) Pick one of the above and prove it from scratch, not using either of the other two.

2. (a) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be measurable. (You can include several equivalent definitions if you wish.)

   (b) Assuming what you need to know about Lebesgue measurable sets in $\mathbf{R}$, prove that, if $\{f_n\}$ is a sequence of measurable functions, then $\limsup f_n$ is also measurable.

3. Suppose f is Lebesgue integrable on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x) \cos(nx) dx = 0$$

   (Hint: What is the simplest type of measurable function?)

4. (a) Define $L^p([0, 1])$ and summarize its important properties (without proof).

   (b) Give an example of a function which is in $L^p([0, 1])$ for $1 \leq p < 3$, but not in $L^3([0, 1])$ and justify your assertions.

5. Evaluate $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} ne^{x^2} \cos(x) \sin^{n-1}(x) dx$, justifying your steps.

6. (a) Define what it means for a function on $[0, 1]$ to be absolutely continuous (AC).

   (b) Prove that an AC function is continuous.

   (c) Prove that an AC function is of bounded variation (BV).

   (d) State (without proof) at least one important consequence of being BV.

7. Let λ be Lebesgue measure and μ be counting measure both regarded as Borel measures on $I = [0, 1]$. Let Δ be the diagonal in $I \times I$; $\Delta = \{(x, y) | x = y\}$.

   (a) Show that Δ is measurable (with respect to the product measure on Borel subsets of $I \times I$).

   (b) Let f be the characteristic function in Δ. Compute the integrals: $\int_I (\int_I f d\lambda) d\mu$, and $\int_I (\int_I f \ d\mu) d\lambda$.

   (c) Explain why the result of part (b) does not contradict Fubini's Theorem. As part of the explanation, you should compute the double integral $\int_{I \times I} f \ d\mu \times d\lambda$.