University at Albany

Department of Mathematics and Statistics

Preliminary Examination

Probability

August, 2005

Do as many problems as possible!

1.a. Suppose X is a random variable with $P(X = 1) = P(X = 0) = P(X = -1) = 1/3$. Find the characteristic function of X.

b. Suppose $P(X_i = 1) = P(X_i = 0) = P(X_i = -1) = 1/3$ for $i = 1, \ldots, n$. Also suppose $X_1, \ldots, X_n$ are independent. Let $S_n = \sum_{i=1}^{n} X_i$. Find the characteristic function of S_n.

c. With the notation of part b and $\sigma^2 = Var(X_i)$, find the characteristic function of $S_n/\sqrt{n\sigma^2}$, and find (with justification) the limit of this function as $n \to \infty$.

d. What does the limit in part c tell you? (Give an explanation rather than a complete proof.)

2.a. Suppose the events A_n are independent. Prove the following statement.

$\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n \text{ infinitely often}) = 1$.

b. Suppose that the events A_n are not necessarily independent. Does the statement to be proved in part a still hold? Justify your response.

3. Let X_n be a sequence of random variables in some probability space, and let a be a constant.

a. Define what it means for X_n to converge to a in probability and what it means for X_n to converge to a almost surely.

b. Give, with justification, an example where X_n satisfies precisely one of the following two properties:

i. X_n converges to 0 in probability.

ii. X_n converges to 0 almost surely.

4. Let $F(x)$ be the distribution function of a random variable. Prove or disprove the following statement.

The number of points where F is not continuous is either finite or countably infinite.

5. You play a game as follows:

A. On the first round, toss a fair coin. If it comes up heads, you lose 1 point and let $X_1 = -1$. Otherwise you win 1 point and let $X_1 = 1$.

B. For the nth round (if $n > 1$), you do nothing if $X_{n-1} \leq 0$ or you did nothing on the $(n-1)$st round. There you let $X_n = X_{n-1}$. Otherwise you do nothing with probability 1/3 and let $X_n = X_{n-1}$. In the remaining circumstance, you do the following. Toss a fair coin. If it comes up heads, you lose 1 point and let $X_n = X_{n-1} - 1$. Otherwise you win 1 point and let $X_n = X_{n-1} + 1$.

What is $E(X_n)$? Justify your answer. Ideally your answer should mention the name of a property which X_n satisfies.

6. Prove or disprove: Suppose X_n converges in probability to X. Then $E(X_n) \to E(X)$ as $n \to \infty$.