Complex Analysis Preliminary Exam

June 4, 1999

1. Let f be a complex-valued harmonic function in a domain $\Omega \subset \mathbf{C}$. Prove that if $|f| = \text{const}$ in Ω, then $f = \text{const}$.

2. Let f be a holomorphic function in the unit disk which is continuous up to the boundary of the disk $\mathbf{T} = \partial\Delta = \{z \in \mathbf{C} : |z| = 1\}$. Prove that if $|f(z)| = 1$ for all $z \in \mathbf{T}$, then f is a rational function.

3. Let f be an entire function such that $\mathrm{Re}(f(z)) \le 0$ for all $z \in \mathbf{C}$. Prove that $f = \text{const}$.

4. For each real t compute the integral $\varphi(t) = \displaystyle\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx$.

5. Construct a conformal mapping of the unit disk onto the crescent
$$\{z \in \mathbf{C} : |z| < 1, \quad \left|z - \frac{1}{2}\right| \ge \frac{1}{2}\} \ .$$

6. How many complex solutions does the equation
$$z = \cos z$$
have? Justify your answer.

   Hint. Use the following fact:

   If an entire function $F(z)$ has no zeros and satisfies
$$|F(z)| \le C_1 e^{C_2|z|} \quad (z \in \mathbf{C})$$
then $F(z) = e^{az+b}$.

7. Let f be a bounded analytic function in the right half-plane. Prove that if
$$f(n) = 0 \text{ for } n = 1, 2, 3, \dots \ ,$$
then $f \equiv 0$.

8. Let f_1, f_2 be entire functions, and let J be the set of all combinations
$$A_1 f_1 + A_2 f_2,$$
where A_1 and A_2 are entire functions. Show that there exists an entire function f such that J consist of all entire functions Af, where A is entire.

   Hint: Use the result of Problem #9.

9. Let $\{a_n\}$ be a sequence in $\mathbf{C}$, $\lim\limits_{n\to\infty} a_n = \infty$. Prove that for any sequence $\{b_n\}$ of complex numbers there exists an entire function f such that $f(a_n) = b_n$.