Preliminary Examination

Complex Analysis

January 1998

1. Suppose that f is an entire function and $f(\mathbf{C}) \cap \{w : \text{Re } w = 0\} = \emptyset$. Prove that f is constant.

2. Evaluate $\displaystyle\int_0^\pi \frac{d\theta}{2+\cos\theta}$.

3. Give an example of a function f which is holomorphic in $\mathbf{C}\setminus\{z_0\}$ for some $z_0 \neq 0$, has an essential singularity at z_0 and is continuous in $\{z : |z| \leq |z_0|\}$. Show that the function given actually has these properties.

4. Find the maximum value of $|g(z)|$ if $g(z) = \dfrac{z}{4z^2-1}$ and z varies over the region $\{z : |z| \geq 1\}$.

5. A. State carefully the Riemann Mapping Theorem.

   B. Let $D = \{z : |z| < 1\}$, $\Omega = \{z : \text{Re } z > 0\}$ and fix $\alpha \in \Omega$. Find all conformal maps g from Ω onto D such that $g(\alpha) = 0$.

6. Suppose that f is a holomorphic function in an open disk D, f is continuous on $\overline{D}$ and $|f|$ is constant and nonzero on ∂D. Prove that f is a rational function.

7. Let P be a nonzero polynomial. Suppose that $\displaystyle\int_{|z|=r} \frac{1}{P(z)} dz \neq 0$ whenever $r > 0$ and the integral is defined. Show that $\deg P = 1$.

8. Suppose that f is an entire function, and for $r > 0$ let $M_f(r) = \sup\{|f(z)| : |z| \leq r\}$. Assume that $0 < \alpha < 1$ and let

$$L(\alpha) = \lim_{r\to\infty} \frac{M_f(\alpha r)}{M_f(r)} \ .$$

   (a) Determine $L(\alpha)$ in the case f is a polynomial.

   (b) Show that $L(\alpha) = 0$ if f is not a polynomial.