Complex Analysis Preliminary Examination

June 1995

$$U = \{z : |z| < 1\},\ \overline{U} = \{z : |z| \le 1\},\ \Gamma = \{|z| = 1\}$$

1. A complex-valued function is said to be harmonic if both its real and imaginary parts are harmonic. Suppose both $f(z)$ and $z\ f(z)$ are complex-valued harmonic functions on a region R. Show that $f(z)$ is analytic on R.

2. Suppose f is entire and the range of f fails to meet some circle. What can be said about f? Justify your answer.

3. Present a contour integration argument to show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \frac{\pi}{e}$$

4. Let f be holomorphic in a neighborhood of $\overline{U}$. Let $f(z) = \sum_{k=0}^{\infty} a_k\ z^k$ and $s_n(z) = \sum_{k=0}^{n} a_k\ z^k$. Show that, among all polynomials P of degree n or less, the integral $\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}) - p(e^{i\theta})|^2\, d\theta$ attains its minimum for $p = s_n$.

5. A. State Cauchy's integral formula for an open disk.

   B. Let f be in the disk algebra, i.e., let f be continuous on $\overline{U}$ and holomorphic on U. Use part (A) to prove that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\, dw \text{ for } z \in U\ .$$

6. Let f be holomorphic in a neighborhood of 0 and suppose that the series $\sum_{k=0}^{\infty} f^{(k)}(0)$ converges. Show that f can be extended to an entire function.

7. Let f be meromorphic in a neighborhood of $\overline{U}$ with no poles on Γ. Let $|A| > \max_{z\in\Gamma} |f(z)|$. Prove that in U, counting multiplicity and order, the number of solutions to $f(z) = A$ equals the number of poles of f.

8. Let $\mathcal{F}$ denote the family of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which are holomorphic in U and satisfy $|a_n| \leq n$ for $n = 0, 1, 2, \ldots$. Prove that any sequence in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of U.

9. Let $f(z)$ be nonconstant and holomorphic in a neighborhood of $\overline{U}$ with $f(0) = a_0$. Let $M = \max_{z\in\Gamma} |f(z)|$. Let $\lambda \in U$ and suppose $f(\lambda) = 0$. Prove that $|\lambda| \geq \frac{|a_0|}{M}$.
[Hint: Consider $g(z) = f\left(\frac{z+\lambda}{1+\overline{\lambda}\, z}\right)$.]