Preliminary Examination in Complex Analysis

January 1995

Let $\mathbf{C}$ be the set of complex numbers and let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$.

1. Let $U : \mathbf{D} \to \mathbf{D}$ be harmonic and $f : \mathbf{D} \to \mathbf{D}$ be analytic. Prove or disprove the following:
   (1) $f \circ U$ is harmonic.
   (2) $U \circ f$ is harmonic.

2. Show that there exists an unbounded analytic function f on $\mathbf{D}$ such that
$$\int_{\mathbf{D}} |f'(z)|^2 \, dA(z) < +\infty,$$
where dA is area measure on $\mathbf{D}$.

3. Suppose f is analytic in $\mathbf{D} - \{0\}$ and unbounded near $z = 0$. If the function $|z|^{\sqrt{2}} f(z)$ is bounded at $z = 0$, show that
$$\lim_{z \to 0} |z|^{\sqrt{2}} f(z) = 0 \quad \text{and} \quad \lim_{z \to 0} |z|^{\sqrt{2}/2} f(z) = \infty.$$

4. Let X be the space of analytic functions f in $\mathbf{D}$ such that
$$\|f\| = \sup_{z \in \mathbf{D}} (1 - |z|^2)|f(z)| < +\infty.$$
If $\{f_n\}$ is a sequence of functions in X such that $\|f_n - f_m\| \to 0$ as $n, m \to +\infty$, show that there exists a function $f \in X$ such that $\|f_n - f\| \to 0$ as $n \to +\infty$.

5. Let f be analytic in $\mathbf{D}$. Show that
$$\sup_{z \in \mathbf{D}} (1 - |z|^2)|f'(z)| \le \sup_{z \in \mathbf{D}} |f(z)|.$$

6. Suppose f is analytic in $\mathbf{D}$. For $z \in \mathbf{D}$ and $0 < r < 1 - |z|$ let $B(z, r) = \{w \in \mathbf{D} : |z - w| < r\}$. Show that
$$|f(z)|^\pi \le \frac{1}{\pi r^2} \int_{B(z,r)} |f(w)|^\pi \, dA(w),$$
where dA is area measure on $\mathbf{D}$.

7. Evaluate the integral
$$I = \int_{|z|=\pi} \frac{\sin z}{z \cos z} \, dz.$$

8. Suppose $\{a_n\}$ is a sequence in $\mathbf{D} - \{0\}$ with $\sum (1 - |a_n|) < +\infty$. Show that
$$\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$
converges (uniformly on compact sets) to an analytic function in $\mathbf{D}$.